Representations of Quivers Respecting a Quiver Automorphism and a Theorem of Kac

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

A theorem of Kac on quiver representations states that the dimension vectors of the indecomposables over an algebraically closed field are precisely the positive roots of the associated symmetric Kac-Moody Lie algebra. Moreover, he proves that the number of isomorphism classes of absolutely indecomposables of a given dimension vector over a finite field is an integer polynomial in the size of the field, independent of the orientation of the quiver and invariant under the Weyl group.

We generalise these results to representations respecting an admissible quiver automorphism (ii-representations) and obtain the positive roots of an associated symmetrisable Kac-Moody Lie algebra. We also show that the number of isomorphism classes of absolutely ii-indecomposables over a finite field is a rational polynomial in the size of the field, again independent of the orientation and invariant under the Weyl group.

When the quiver is affine, we calculate these polynomials explicitly and see that the coefficients are all non-negative integers. We relate the constant terms to the weight multiplicities of a certain integrable module for the symmetrisable Kac-Moody Lie algebra, given as the fixed points of a symmetric Kac-Moody Lie algebra under a diagram automorphism.

Finally we consider species of valued quivers over finite fields. To each valued quiver there is naturally associated a quiver with admissible automorphism and the representations of the species can be thought of as the representations for the quiver over a finite field extension such that the actions of the admissible automorphism and the Galois group coincide. Using this, we offer a more representation-theoretic proof of the generalisation of Kac's Theorem to the species setting.

Contents

Chapter 1

Introduction

The study of representations of quivers has its origin in a paper by Gabriel [11], who showed that a connected quiver was of finite representation type if and only if its underlying graph was Dynkin (of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$). In doing so, he observed that there is a bijection between the isomorphism classes of indecomposable representations and the set of positive roots of the associated semisimple Lie algebra. In [2], Bernstein, Gelfand and Ponomarev gave a more direct proof of this result. Namely, they obtained all the indecomposables (up to isomorphism) by applying sequences of reflection functors to the simple representations, in a manner analogous to how the roots are obtained by applying relections in the Weyl group to the simple roots.

This work was extended by Donovan and Freislich [9], and independently by Nazarova [34], to cover quivers of affine, or extended Dynkin, type. They described the set of isomorphism classes of indecomposables, hence proving that these quivers are of tame representation type. A more unified approach, which also works for species of Dynkin or affine type, can be found in [8]. We again have a connection with the root systems of the affine Kac-Moody Lie algebras. That is, the dimension vectors of the indecomposables are precisely the positive roots. This does not extend to a bijection with the isomorphism classes though, since to each imaginary root there corresponds a $\mathbb{P}^1$-family of indecomposables.

Finally, Kac proved in [22] that this correspondence holds in general. That is, for a finite quiver without vertex loops, the set of dimension vectors of the indecomposables over an algebraically closed field is precisely the set of positive roots of the associated (symmetric) Kac-Moody Lie algebra. Kac also proposed a number of conjectures suggesting a deeper relationship between the representations of quivers and the structure of the Kac-Moody Lie algebras. These concern the number of isomorphism classes of absolutely indecomposable representations over finite fields and the multiplicities of the roots in the Lie algebra.

Around the same time, Tanisaki [40] considered representations of quivers with an automorphism. He showed that the only such pairs having finitely many isomorphism classes of indecomposables are when the underlying graph of the quiver is Dynkin. Moreover, the dimension vectors of the indecomposables now correspond to the positive roots of an associated valued graph — in this case, the Dynkin graphs of type $\mathbb{B}$, $\mathbb{C}$, $\mathbb{F}$ and $\mathbb{G}$.

This suggests an alternative approach to that of species in the attempt to generalise Kac's Theorem and thereby obtain the positive root systems of all symmetrisable Kac-Moody Lie algebras — namely by studying pairs consisting of a quiver and an automorphism and the corresponding isomorphically invariant representations (ii-representations). This has the advantage that we can now work over algebraically closed fields and it is this approach that we shall adopt in this thesis.

Recently, Reineke [35] has used the notion of ii-indecomposables (in his language, γ-symmetric indecomposables) to construct the so-called quantic monoid associated to any semisimple Lie algebra $\mathfrak{g}$. The corresponding monoid ring can be thought of as the specialisation to $q = 0$ of the positive part of a twisted form of the quantised enveloping algebra for $\mathfrak{g}$. On the other hand, it can be realised as the monoid of generic extensions of ii-representations, a purely geometric construction.

Some of the results in this thesis now form the content of a paper [20], which is available as a preprint on the Mathematics ArXiv website.

1.1 Quivers and Root Systems

A quiver $\mathcal{Q}$ is a finite directed graph with vertices $\mathcal{I}$ and arrows $\mathcal{A}$. We shall also assume that $\mathcal{Q}$ has no vertex loops. There are two maps $s, t : \mathcal{A} \to \mathcal{I}$ which describe where each arrow starts and terminates. In particular, we write $\rho : s(\rho) \to t(\rho)$. A quiver is connected if its underlying graph is connected.

We associate to $\mathcal{Q}$ a symmetric matrix A indexed by $\mathcal{I}$ as follows. (In the terminology of Section 1.4, A is a symmetric generalised Cartan matrix.)

$$a_{ij} := \begin{cases} 2 & \text{if } i = j; \\ -\#\{\text{edges between } i \text{ and } j\} & \text{if } i \neq j. \end{cases} \tag{1.1.1}$$

Clearly A is independent of the orientation of $\mathcal{Q}$.

The root lattice $\mathbb{Z}\mathcal{I}$ of $\mathcal{Q}$ is the free abelian group on elements e_i for $i \in \mathcal{I}$. We partially order $\mathbb{Z}\mathcal{I}$ by $\alpha = \sum_i \alpha_i e_i \geq 0$ if and only if $\alpha_i \geq 0$ for all $i \in \mathcal{I}$. If $\alpha > 0$, then we define its height to be $\operatorname{ht} \alpha := \sum_i \alpha_i$. We also write

$$\bar{\alpha} := \operatorname{hcf}\{\alpha_i \mid i \in \mathcal{I}\} \tag{1.1.2}$$

and say that α is indivisible if $\bar{\alpha} = 1$.

We endow $\mathbb{Z}\mathcal{I}$ with a symmetric bilinear form $(-,-)$ via

$$(e_i, e_j) := a_{ij} \tag{1.1.3}$$

and then for each vertex i we have the reflection

$$r_i : \alpha \mapsto \alpha - (\alpha, e_i)e_i. \tag{1.1.4}$$

These reflections generate the Weyl group W of $\mathcal{Q}$. Clearly the bilinear form $(-,-)$ is W-invariant. The length $\ell(w)$ of an element $w \in W$ is defined to be the minimum length of an expression $w = r_{i_1} \cdots r_{i_m}$ with $i_j \in \mathcal{I}$.

We can describe the set $\Delta \subset \mathbb{Z}\mathcal{I}$ of roots of $\mathcal{Q}$ combinatorially as follows. We have the simple roots

$$\Pi := \{e_i \mid i \in \mathcal{I}\} \tag{1.1.5}$$

and the fundamental region

$$F := \{\alpha > 0 \mid (\alpha, e_i) \leq 0 \text{ for all } i \in \mathcal{I} \text{ and } \operatorname{supp} \alpha \text{ connected}\}, \tag{1.1.6}$$

where $\operatorname{supp} \alpha$ is the support of α — the full subquiver on those vertices i such that $\alpha_i \neq 0$.

N.B. The fundamental region may well be empty.

The sets of real and imaginary roots are now

$$\Delta^{\text{re}} := W \cdot \Pi \qquad \text{and} \qquad \Delta^{\text{im}} := \pm W \cdot F. \tag{1.1.7}$$

Moreover, each root is either positive or negative and we write Δ_+ for the set of positive roots.

Let α be a root. We note that $(\alpha, \alpha) = 2$ if α is real, whereas $(\alpha, \alpha) \leq 0$ if α is imaginary.

1.2 Representations of Quivers

Let K be a field. A K-representation $X = \{V_i, f_\rho \mid i \in \mathcal{I}, \rho \in \mathcal{A}\}$ of a quiver $\mathcal{Q}$ is given by a finite dimensional vector space V_i for each vertex and a linear map $f_\rho : V_{s(\rho)} \to V_{t(\rho)}$ for each arrow.

If $Y = \{W_i, g_\rho\}$ is another representation of $\mathcal{Q}$, then we define a morphism $\theta : X \to Y$ to be a collection of linear maps $\theta_i : V_i \to W_i$ for each i such that, for each arrow ρ, we have a commutative square

$$\begin{array}{ccc} V_{s(\rho)} & \xrightarrow{f_\rho} & V_{t(\rho)} \\ {\scriptstyle \theta_{s(\rho)}}\downarrow & & \downarrow{\scriptstyle \theta_{t(\rho)}} \\ W_{s(\rho)} & \xrightarrow[g_\rho]{} & W_{t(\rho)} \end{array} \tag{1.2.1}$$

This is an isomorphism if and only if each θ_i is an isomorphism.

The direct sum of two representations is given by

$$X \oplus Y := \{V_i \oplus W_i, f_\rho \oplus g_\rho\} \tag{1.2.2}$$

and a representation is called indecomposable if it is not isomorphic to a proper direct sum of two representations. We obtain an abelian category in which the Krull-Remak-Schmidt Theorem holds.

Theorem 1.2.1. *Every representation is isomorphic to a direct sum of indecomposables and the isomorphism classes and multiplicities of these summands are uniquely determined.*

The dimension vector of a representation $X = \{V_i, f_\rho\}$ is

$$\underline{\dim}\, X := \sum_i (\dim V_i) e_i \in \mathbb{Z}\mathcal{I}. \tag{1.2.3}$$

(In general we shall call any $\alpha \in \mathbb{Z}\mathcal{I}$ with $\alpha \geq 0$ a dimension vector.) By fixing bases, we see that the representations of dimension vector α are parametrised by the affine space

$$\mathrm{Rep}(\mathcal{Q}, \alpha, K) := \prod_{\rho \in \mathcal{A}} \mathbb{M}\left(\alpha_{t(\rho)} \times \alpha_{s(\rho)}, K\right). \tag{1.2.4}$$

We shall identify the points of $\mathrm{Rep}(\mathcal{Q}, \alpha, K)$ with the corresponding representations of $\mathcal{Q}$.

The group

$$\mathrm{GL}(\alpha, K) := \prod_{i \in \mathcal{I}} \mathrm{GL}(\alpha_i, K) \tag{1.2.5}$$

acts on $\mathrm{Rep}(\mathcal{Q}, \alpha, K)$ by conjugation — given $g = (g_i) \in \mathrm{GL}(\alpha, K)$ and $X = (X_\rho) \in \mathrm{Rep}(\alpha, K)$, the ρ-th component of $g \cdot X$ is

$$(g \cdot X)_\rho := g_{t(\rho)} X_\rho g_{s(\rho)}^{-1}. \tag{1.2.6}$$

There is a 1–1 correspondence between the isomorphism classes of K-representations of dimension vector α and the $\mathrm{GL}(\alpha, K)$-orbits on $\mathrm{Rep}(\mathcal{Q}, \alpha, K)$.

1.3 Path Algebras

For a field K and a quiver $\mathcal{Q}$, we recall the definition of the path algebra $K\mathcal{Q}$ and the equivalence between the category of finite dimensional $K\mathcal{Q}$-modules and the category of K-representations of $\mathcal{Q}$. These results can be found in [1], Chapter III.

A path in $\mathcal{Q}$ is a sequence of arrows $p = \rho_m \cdots \rho_1$ such that $t(\rho_i) = s(\rho_{i+1})$. We set $s(p) = s(\rho_1)$ and $t(p) = t(\rho_m)$. We also have the trivial path ε_i for each vertex i. The path algebra $K\mathcal{Q}$ has basis the paths in $\mathcal{Q}$ and with $q \cdot p$ being the concatenation of the two paths if $t(p) = s(q)$ and 0 otherwise.

We note that $K\mathcal{Q}$ is a finitely generated associative algebra and that the ε_i for $i \in \mathcal{I}$ give a complete set of primitive idempotents.

Suppose that $X = \{V_i, f_\rho\}$ is a K-representation of $\mathcal{Q}$. Let $V := \bigoplus_i V_i$ and write π_i and η_i for the maps $V \twoheadrightarrow V_i$ and $V_i \hookrightarrow V$ respectively. We obtain a $K\mathcal{Q}$-module $\mathcal{X}$ with underlying vector space V via

$$\varepsilon_i x := \eta_i \pi_i(x) \qquad \text{and} \qquad \rho x := \eta_{t(\rho)} f_\rho \pi_{s(\rho)}(x) \qquad \text{for all } x \in V. \tag{1.3.1}$$

Conversely, if $\mathcal{X}$ is a finite dimensional $K\mathcal{Q}$-module with underlying vector space V, then we obtain a K-representation of $\mathcal{Q}$ by setting $V_i := \varepsilon_i V$ and $f_\rho(x) = \rho x$.

These constructions extend to functors and hence we get

Proposition 1.3.1. *The categories of finite dimensional $K\mathcal{Q}$-modules and K-representations of $\mathcal{Q}$ are equivalent.*

We also note that the path algebra has a natural grading by path length $K\mathcal{Q} = \bigoplus_{n \geq 0} \Lambda_n$, where each ε_i has length 0 and the path $p = \rho_m \cdots \rho_1$ has length m. The space Λ_0 with basis the trivial paths is a basic semisimple subalgebra and Λ_1, which has basis the arrows of $\mathcal{Q}$, is a Λ_0-bimodule.

Proposition 1.3.2. *The path algebra $K\mathcal{Q}$ is isomorphic to the tensor algebra $T(\Lambda_0, \Lambda_1)$.*

1.4 Kac-Moody Lie Algebras

In this section we recall the definition and some of the basic properties of a symmetrisable Kac-Moody Lie algebra over $\mathbb{C}$ (see [25]).

A generalised Cartan matrix (GCM) is a square matrix C with 2 on the leading diagonal, all other entries non-positive integers and such that $c_{ij} = 0$ if and only if $c_{ji} = 0$. The matrix C is called symmetrisable if there exists an invertible diagonal matrix D such that DC is symmetric. If C is symmetrisable, then we can always assume D has non-negative integer entries.

We have seen that every graph gives rise to a symmetric GCM, and this correspondence is actually a bijection. Similarly, there is a bijection between symmetrisable GCMs and valued graphs: if C is a symmetrisable GCM indexed by $\mathcal{I}$, then we obtain a valued graph as follows. We take $\mathcal{I}$ as the vertex set and draw a valued edge

$$i \xrightarrow{(|c_{ji}|,|c_{ij}|)} j \tag{1.4.1}$$

whenever $c_{ij} \neq 0$.

A symmetrisable GCM C is called indecomposable if its corresponding valued graph is connected.

Let C be a symmetrisable GCM of size $m = |\mathcal{I}|$ and rank l and fix a decomposition $C = D^{-1}B$, where D and B are both integer valued. Let $\mathfrak{h}$ be a $2m-l$ dimensional complex vector space and pick linearly independent elements $H_i \in \mathfrak{h}$ and $e_i \in \mathfrak{h}^*$ for $i \in \mathcal{I}$ such that $e_j(H_i) = c_{ij}$. The triple $(\mathfrak{h}, \{H_i\}, \{e_i\})$ is called a (minimal) realisation of C. We write $\mathfrak{h}'$ and Λ for the subspaces of $\mathfrak{h}$ and $\mathfrak{h}^*$ spanned by the H_i and e_i respectively.

Lemma 1.4.1. *Any two realisations $(\mathfrak{h}^1, \{H_i^1\}, \{e_i^1\})$ and $(\mathfrak{h}^2, \{H_i^2\}, \{e_i^2\})$ of C are isomorphic, in the sense that there is an isomorphism $\phi : \mathfrak{h}^1 \to \mathfrak{h}^2$ such that $\phi(H_i^1) = (H_i^2)$ and $\phi^*(e_i^2) = e_i^1$.*

The Kac-Moody Lie algebra $\mathfrak{g}$ associated to a realisation $(\mathfrak{h}, \{H_i\}, \{e_i\})$ of C is defined to be the Lie algebra generated by $\mathfrak{h}$ and elements E_i, F_i for $i \in \mathcal{I}$ subject to the following Serre relations:

$$\begin{aligned} &[H, H'] = 0, && [H, E_j] = e_j(H)E_j, && (\operatorname{ad} E_i)^{1-c_{ij}} E_j = 0, \\ &[E_i, F_j] = \delta_{ij} H_i, && [H, F_j] = -e_j(H)F_j, && (\operatorname{ad} F_i)^{1-c_{ij}} F_j = 0, \end{aligned} \tag{1.4.2}$$

where $H, H' \in \mathfrak{h}$.

The elements E_i, F_i, H_i are called the Chevalley generators and these generate the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The centre $\mathfrak{c}$ (of $\mathfrak{g}$ or $\mathfrak{g}'$) is given by those $H \in \mathfrak{h}$ such that $e_i(H) = 0$ for all i. This has dimension $r = m - l$, the corank of C.

For $\alpha \in \mathfrak{h}^*$ we define

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [H, x] = \alpha(h)x \text{ for all } H \in \mathfrak{h}\}. \tag{1.4.3}$$

We denote by Δ the set of non-zero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$. Then $\Delta \subset \Lambda$ and is called the set of roots of $\mathfrak{g}$. The number $\dim \mathfrak{g}_\alpha$ is called the multiplicity of α. We also note that $\mathfrak{g}_0 = \mathfrak{h}$.

If C is symmetric, corresponding to the graph $\mathcal{Q}$, then the set of roots coincides with the combinatorial description given in Section 1.1. More generally, if C corresponds to the valued graph Γ, then we can again define the root system combinatorially as follows.

As before, the root lattice $\mathbb{Z}\mathcal{I}$ of Γ is the free abelian group with generators the simple roots e_i for $i \in \mathcal{I}$. We endow this with the symmetric bilinear form $(-,-)$ determined by B — namely

$$(e_i, e_j) := b_{ij}. \tag{1.4.4}$$

N.B. If Γ is connected, then this bilinear form is unique up to a scalar.

The reflections r_i of $\mathbb{Z}\mathcal{I}$ are defined by

$$r_i : \alpha \mapsto \alpha - \frac{1}{d_i}(\alpha, e_i)e_i \tag{1.4.5}$$

and these generate the Weyl group W.

The real roots Δ^{re} are the images under W of the simple roots and the imaginary roots Δ^{im} are $\pm W \cdot F$, where F is the fundamental region

$$F := \{\alpha > 0 \mid (\alpha, e_i) \leq 0 \text{ for all } i \text{ and } \operatorname{supp}\alpha \text{ connected}\}. \tag{1.4.6}$$

We have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{1.4.7}$$

where $\mathfrak{n}_\pm := \oplus_{\alpha>0}\mathfrak{g}_{\pm\alpha}$. Moreover, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are generated by the E_i and F_i respectively.

We can equip $\mathfrak{h}$ with a non-degenerate symmetric bilinear form $(-,-)$ such that $(H_i, H) = \frac{1}{d_i}e_i(H)$ for all $H \in \mathfrak{h}$. In particular, $(H_i, H_j) = \frac{1}{d_i d_j}b_{ij}$.

Lemma 1.4.2. *Any two non-degenerate symmetric bilinear forms on $\mathfrak{h}$ satisfying $(H_i, H) = \frac{1}{d_i}e_i(H)$ differ by an isomorphism fixing $\mathfrak{h}'$.*

Proof.
We know that $(-,-)$ is non-degenerate on $\mathfrak{h}$ and that the kernel of $(-,-)$ restricted to the subspace $\mathfrak{h}'$ is the centre $\mathfrak{c}$. Fix a complement V to $\mathfrak{c}$ in $\mathfrak{h}'$ and let $z_1 \dots, z_r$ and $v_1, \dots, v_l$ be bases for $\mathfrak{c}$ and V respectively. Then (see for example [28], Chapter XV, Lemma 10.1) there exist elements $x_1, \dots, x_r$ in $\mathfrak{h}$ such that

$$(x_i, z_j) = \delta_{ij}, \qquad (x_i, v_j) = 0 \qquad \text{and} \qquad (x_i, x_j) = 0. \tag{1.4.8}$$

In particular, the z_i, v_j and x_k form a basis for $\mathfrak{h}$.

Now suppose that $(-,-)_1$ and $(-,-)_2$ are two non-degenerate bilinear forms on $\mathfrak{h}$ satisfying $(H_i, H_j) = \frac{1}{d_i d_j}b_{ij}$. We can take $x_1, \dots, x_r$ and $y_1, \dots, y_r$ satisfying (1.4.8) for $(-,-)_1$ and $(-,-)_2$ respectively and define an automorphism ϕ of $\mathfrak{h}$ by $H_i \mapsto H_i$ and $x_i \mapsto y_i$. Then $(\phi(H), \phi(H'))_2 = (H, H')_1$ for all $H, H' \in \mathfrak{h}$.
■

The bilinear form determines a bijection $\nu : \mathfrak{h} \to \mathfrak{h}^*$ sending H_i to $\frac{1}{d_i}e_i$ and hence induces a bilinear form on $\mathfrak{h}^*$. This satisfies $(e_i, e_j) = b_{ij}$ and thus we recover the original form on $\mathbb{Z}\mathcal{I}$ defined by (1.4.4).

The Chevalley involution ω of $\mathfrak{g}$ is given by

$$\omega(E_i) = -F_i, \qquad \omega(F_i) = -E_i \qquad \omega(H) = -H. \tag{1.4.9}$$

This satisfies $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.

Given a bilinear form $(-,-)$ on $\mathfrak{h}$ as above, we can extend it uniquely to an invariant non-degenerate symmetric bilinear form on the whole of $\mathfrak{g}$. This satisfies

$$(E_i, F_i) = \frac{1}{d_{\mathbf{i}}}. \tag{1.4.10}$$

Moreover, if $\alpha \in \Delta$ and $x_\pm \in \mathfrak{g}_{\pm\alpha}$, then

$$[x_+, x_-] = (x_+, x_-)\nu^{-1}(\alpha). \tag{1.4.11}$$

We shall also need the following result on ideals of $\mathfrak{g}$.

Proposition 1.4.3. *The Lie algebra $\mathfrak{g}$ is simple if and only its GCM C is non-degenerate and indecomposable. Conversely, if C is indecomposable, then every ideal of $\mathfrak{g}$ either contains $\mathfrak{g}'$ or is contained in $\mathfrak{c}$.*

1.5 Kac's Theorem and Conjectures

Let K be an algebraically closed field. Then the subset of indecomposable representations $\mathrm{Ind}(\alpha, K) \subset \mathrm{Rep}(\alpha, K)$ is a constructible set, stable under $\mathrm{GL}(\alpha, K)$. As in Appendix B we can therefore define the number of parameters

$$\mu(\alpha, K) := \dim_{\mathrm{GL}(\alpha,K)} \mathrm{Ind}(\alpha, K) \tag{1.5.1}$$

and the number of top-dimensional families of orbits

$$t(\alpha, K) := \mathrm{top}_{\mathrm{GL}(\alpha,K)} \mathrm{Ind}(\alpha, K). \tag{1.5.2}$$

We can now state Kac's Theorem. This was first proved in [22, 23] but see also [24, 27].

Theorem 1.5.1 (Kac). *Let $\mathcal{Q}$ be a quiver and K an algebraically closed field.*

1. *The dimension vectors of the indecomposable representations are precisely the positive roots Δ_+;*
2. *For $\alpha \in \Delta_+$ we have $\mu(\alpha, K) = 1 - \frac{1}{2}(\alpha, \alpha)$ and $t(\alpha, K) = 1$.*

In particular, there is a unique isomorphism class of indecomposables of dimension vector α if and only if α is a positive real root. We also note that the dimension vectors, number of parameters and number of top-dimensional families of orbits are all independent of both the orientation of $\mathcal{Q}$ and the characteristic of the field K.

Now suppose that $\mathbb{F}_q$ is a finite field with q elements and write K for its algebraic closure. An $\mathbb{F}_q\mathcal{Q}$-module X is called absolutely indecomposable if the $K\mathcal{Q}$-module $K \otimes_{\mathbb{F}_q} X$ is indecomposable. If $\underline{\dim}\, X = \alpha$, then this is equivalent to saying that X corresponds to a point in the set $\mathrm{Ind}(\alpha, K) \cap \mathrm{Rep}(\alpha, \mathbb{F}_q)$, where we have identified $\mathrm{Rep}(\alpha, \mathbb{F}_q)$ with the set of points of $\mathrm{Rep}(\alpha, K)$ having each co-ordinate in $\mathbb{F}_q$.

We denote the number of isomorphism classes of absolutely indecomposable $\mathbb{F}_q\mathcal{Q}$-modules of dimension vector α by $A(\alpha, q)$. The following is proved in [23] (see also [19, 24]).

Theorem 1.5.2. *The numbers $A(\alpha, q)$ are polynomials in q with integer coefficients. For $\alpha \in \Delta_+$ this polynomial has degree $\mu(\alpha) = 1 - \frac{1}{2}(\alpha, \alpha)$ and leading coefficient $t(\alpha) = 1$. Moreover, $A(\alpha, q)$ is independent of the orientation of $\mathcal{Q}$ and is invariant under the action of the Weyl group.*

Regarding these polynomials, Kac made the following conjectures.

Conjecture 1. The coefficients of $A(\alpha, q)$ are all non-negative integers.

Conjecture 2. The constant term $A(\alpha, 0)$ equals $\dim \mathfrak{g}(\mathcal{Q})_\alpha$, the multiplicity of α viewed as a root of the Kac-Moody Lie algebra $\mathfrak{g}(\mathcal{Q})$.

These are known to be true when $\mathcal{Q}$ is a Dynkin or affine quiver using the classification of the isomorphism classes of indecomposables. Recently, both conjectures have been shown to hold when α is an indivisible root [5].

1.6 The Main Theorem

The main result of this thesis is the following theorem generalising Kac's Theorem, Theorem 1.5.1. The terminology and notation is taken from Chapter 2.

Theorem 1.6.1. *Let $(\mathcal{Q}, \mathbf{a})$ be a quiver with admissible automorphism of order n. Let Γ be the associated valued graph and K an algebraically closed field of characteristic not dividing n.*

1. *The images under f of the dimension vectors of the ii-indecomposables for $(\mathcal{Q}, \mathbf{a})$ are precisely the positive roots $\Delta(\Gamma)_+$ of Γ;*
2. *If $\alpha \in \Delta(\Gamma)_+^{\mathrm{re}}$ is a positive real root, then there exists a unique isomorphism class of ii-indecomposables having dimension vector $f^{-1}(\alpha)$.*

Here $f : (\mathbb{Z}\mathcal{I})^{\langle \mathbf{a} \rangle} \to \mathbb{Z}\mathbf{I}$ is the canonical bijection between the points of the root lattice for $\mathcal{Q}$ fixed by $\mathbf{a}$ and the root lattice for Γ.

As before, we see that the dimension vectors of the ii-indecomposables are independent of both the orientation of $\mathcal{Q}$ and the characteristic of the field K. We note, however, that the converse of Part 2 does not hold in general (c.f. the remark following Theorem 1.5.1). That is, there may exist imaginary roots α for which there is a unique isomorphism class of ii-indecomposables having dimension vector $f^{-1}(\alpha)$. We exhibit such an example in Section 3.7.

We also prove a theorem analogous to Theorem 1.5.2. Let $(\mathcal{Q}, \mathbf{a})$ be a quiver with admissible automorphism of order n and let Γ be the associated valued graph. We write $A\text{-}ii(\alpha, q)$ for the number of isomorphism classes of absolutely ii-indecomposable representations of dimension vector $f^{-1}(\alpha)$ defined over the finite field $\mathbb{F}_q$.

Theorem 1.6.2. *The numbers $A\text{-}ii(\alpha, q)$ for $q \equiv 1 \bmod n$ are polynomials in q with rational coefficients, independent of the orientation of $\mathcal{Q}$ and invariant under the action of the Weyl group $W(\Gamma)$. Moreover, the coefficients have denominators bounded by $n\bar{\alpha}$.*

Let K be an algebraically closed field and write $\text{Ind-ii}(\alpha, K)$ for the constructible subset of $\text{Rep}(f^{-1}(\alpha), K)$ corresponding to the ii-indecomposables. The affine algebraic group $\mathrm{GL}(f^{-1}(\alpha), K)$ acts on $\mathrm{Ind\text{-}ii}(\alpha, K)$ and again there is a bijection between the orbits of this group action and the isomorphism classes of ii-indecomposables over K. Therefore we can consider the number of parameters

$$\mu\text{-}ii(\alpha, K) := \dim_{\mathrm{GL}(f^{-1}(\alpha),K)} \mathrm{Ind\text{-}ii}(\alpha, K) \tag{1.6.1}$$

and the number of top-dimensional families of orbits

$$t\text{-}ii(\alpha, K) := \mathrm{top}_{\mathrm{GL}(f^{-1}(\alpha),K)} \mathrm{Ind\text{-}ii}(\alpha, K). \tag{1.6.2}$$

Using Theorem 1.6.2 and some properties of affine schemes of finite type over the integers we deduce the following.

Corollary 1.6.3. *Let K be an algebraically closed field of characteristic not dividing n. Then the numbers $\mu\text{-}ii(\alpha, K)$ and $t\text{-}ii(\alpha, K)$ are given by the degree and leading coefficient of the polynomial $A\text{-}ii(\alpha, q)$. In particular, they are also independent of the orientation of $\mathcal{Q}$ and of the characteristic of the field K.*

We also generalise Kac's two conjectures. To do this, we first consider the dual quiver with automorphism $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$. (The terminology comes from the fact that the associated valued graph is dual to Γ, see Sections 3.3 and 3.4.)

Let $\mathfrak{g}$ denote the Kac-Moody Lie algebra corresponding to $\widetilde{\mathcal{Q}}$. Then the automorphism $\tilde{\mathbf{a}}$ naturally induces an automorphism of the derived Lie algebra $\mathfrak{g}'$ and $\mathfrak{g}(\Gamma)'$ embeds in the fixed-point subalgebra. In fact, we show in Chapter 7 that this automorphism can be extended to the whole of $\mathfrak{g}$ such that $\mathfrak{g}(\Gamma)$ embeds in the fixed-point algebra $\mathcal{L}$. We can now view $\mathcal{L}$ as an integrable $\mathfrak{g}(\Gamma)$-module and we prove that the set of non-zero weights is precisely the set of roots $\Delta(\Gamma)$.

Conjecture 1′. The coefficients of $A\text{-}ii(\alpha, q)$ are all non-negative integers.

Conjecture 2′. The constant term $A\text{-}ii(\alpha, 0)$ equals $\dim \mathcal{L}_\alpha$, the multiplicity of α viewed as a weight of the integrable $\mathfrak{g}(\Gamma)$-module $\mathcal{L}$.

We prove that these conjectures hold when α is a positive real root, for then $A\text{-}ii(\alpha, q) = 1 = \dim \mathcal{L}_\alpha$. In the case when $\mathcal{Q}$ is affine, we also calculate $A\text{-}ii(\alpha, q)$ and $\dim \mathcal{L}_\alpha$ for all imaginary roots $\alpha = m\delta_\Gamma$ of Γ and show that the conjectures hold.

We now describe briefly the organisation of this thesis. In Chapter 2 we introduce the basic notation and terminology. We prove some combinatorial lemmas relating the structure of the fixed points of the root lattice for $\mathcal{Q}$ to that of the root lattice for Γ. We also prove Part 2 and one direction of Part 1 of Theorem 1.6.1.

In Chapter 3 we consider the relationship to skew group algebras and introduce the dual quiver with automorphism $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$. Using this, we complete the proof of Theorem 1.6.1. We also offer a counter-example to the converse of Part 2.

Chapter 4 is concerned with the numbers $A\text{-}ii(\alpha, q)$. We first factorise the generating function for the total number of isomorphism classes of ii-representations over a finite field and show that there is a factor corresponding to each divisor of n (or subgroup of $\langle \mathbf{a} \rangle$). This allows us to derive a formula for the numbers $A\text{-}ii(\alpha, q)$ and prove Theorem 1.6.2.

In Chapter 5 we apply this result to prove Corollary 1.6.3 as described above and in Chapter 6 we consider the affine quivers and their automorphisms, computing the numbers $A\text{-}ii(m\delta_\Gamma, q)$ in each case.

We then discuss diagram automorphisms of Kac-Moody Lie algebras in Chapter 7 and in particular the automorphisms of the affine algebras. In this way, we prove Conjectures $1'$ and $2'$ for the affine quivers.

Finally, in Chapter 8, we discuss the connection to species over finite fields and offer a representation-theoretic proof of the generalisation of Kac's Theorem to species.

There are also two appendices. The first contains all the results on affine schemes that we require in Chapter 5 and the second discusses some aspects of affine algebraic group actions on affine varieties.

Chapter 2

Quivers With An Automorphism

In this chapter we introduce the notion of an admissible automorphism of a quiver and explain how this gives rise to a symmetrisable GCM, and hence a valued graph. We show that there are natural connections between the combinatorial properties of the quiver which respect the automorphism and those of the valued graph.

The automorphism also acts functorially on the category of all representations and this determines an action on the set of isomorphism classes. We define the isomorphically invariant representations, or ii-representations, to be those representations corresponding to the fixed points of this latter action.

In this way we obtain an additive subcategory. The ii-indecomposables are now taken to be the indecomposable objects in this subcategory.

2.1 Quivers with an Admissible Automorphism

Let $\mathcal{Q}$ be a quiver, which we will assume is finite and without vertex loops. An admissible automorphism $\mathbf{a}$ of $\mathcal{Q}$ is a quiver automorphism such that no arrow connects two vertices in the same orbit. This notion was first introduced by Lusztig [30] where, given such a pair $(\mathcal{Q}, \mathbf{a})$, he describes how to construct a symmetric matrix B indexed by the vertex orbits $\mathbf{I}$. Namely

$$b_{\mathbf{ij}} := \begin{cases} 2\#\{\text{vertices in } \mathbf{i}\text{-th orbit}\} & \text{if } \mathbf{i} = \mathbf{j}; \\ -\#\{\text{edges between } \mathbf{i}\text{-th and } \mathbf{j}\text{-th orbits}\} & \text{if } \mathbf{i} \neq \mathbf{j}. \end{cases} \tag{2.1.1}$$

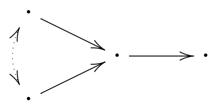
Let

$$d_{\mathbf{i}} := b_{\mathbf{ii}}/2 = \#\{\text{vertices in } \mathbf{i}\text{-th orbit}\} \tag{2.1.2}$$

and set $D = \operatorname{diag}(d_{\mathbf{i}})$. Then $C = D^{-1}B$ is a symmetrisable GCM and we write Γ for the corresponding valued graph. That is, Γ has vertex set $\mathbf{I}$ and we draw an edge $\mathbf{i} - \mathbf{j}$ equipped with the ordered pair $(|c_{\mathbf{ji}}|, |c_{\mathbf{ij}}|)$ whenever $c_{\mathbf{ij}} \neq 0$.

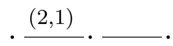
We know from [30], Proposition 14.1.2 that every symmetrisable GCM (or valued graph) can be obtained from such a pair $(\mathcal{Q}, \mathbf{a})$, though this pair will not be unique. We also note that given a graph with an admissible automorphism, we can always assign a compatible orientation to obtain a quiver with automorphism.

Example 2.1.1. *Consider the quiver $\mathbb{D}_4$ with automorphism*



The corresponding symmetrisable GCM is then

$$\begin{pmatrix} 2 & -1 & \\ -2 & 2 & -1 \\ & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & & \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 & -2 & \\ -2 & 2 & -1 \\ & -1 & 2 \end{pmatrix}$$

and the valued graph Γ is of type $\mathbb{C}_3$ 

Using the decomposition $C = D^{-1}B$ we can endow the root lattice $\mathbb{Z}\mathbf{I}$ of Γ with a symmetric bilinear form $(-,-)_\Gamma$, as in Section 1.4. Namely, we set

$$(e_{\mathbf{i}}, e_{\mathbf{j}})_\Gamma := b_{\mathbf{ij}}. \tag{2.1.3}$$

We also recall the Weyl group $W(\Gamma)$, generated by the reflections

$$r_{\mathbf{i}} : \alpha \mapsto \alpha - \frac{1}{d_{\mathbf{i}}}(\alpha, e_{\mathbf{i}})_\Gamma e_{\mathbf{i}} \quad \text{for all } \alpha \in \mathbb{Z}\mathbf{I}, \tag{2.1.4}$$

and the root system $\Delta(\Gamma)$. The real roots $\Delta(\Gamma)^{\mathrm{re}}$ are the images under $W(\Gamma)$ of the simple roots $e_{\mathbf{i}}$ whereas the set of imaginary roots $\Delta(\Gamma)^{\mathrm{im}}$ is $\pm W(\Gamma)\cdot F_\Gamma$, where F_Γ is the fundamental region

$$F_\Gamma := \{\alpha > 0 \mid (\alpha, e_{\mathbf{i}})_\Gamma \leq 0 \text{ for all } \mathbf{i} \text{ and } \operatorname{supp}\alpha \text{ connected}\}. \tag{2.1.5}$$

The automorphism $\mathbf{a}$ acts naturally on the root lattice $\mathbb{Z}\mathcal{I}$ for $\mathcal{Q}$ by $\mathbf{a}(e_i) := e_{\mathbf{a}(i)}$ and the bilinear form $(-,-)_{\mathcal{Q}}$ is $\mathbf{a}$-invariant. (In other words, $\mathbf{a}$ induces an automorphism of the GCM A, using the terminology of Proposition 7.1.2.) We have a canonical bijection

$$f : (\mathbb{Z}\mathcal{I})^{\langle \mathbf{a}\rangle} \to \mathbb{Z}\mathbf{I} \tag{2.1.6}$$

from the fixed points in the root lattice of $\mathcal{Q}$ to the root lattice of Γ. This is given by $f(\alpha)_{\mathbf{i}} := \alpha_i$ for any vertex i in the $\mathbf{i}$-th orbit.

The admissibility of $\mathbf{a}$ implies that the reflections r_i and r_j commute whenever i and j lie in the same $\mathbf{a}$-orbit. Therefore the element

$$s_{\mathbf{i}} := \prod_{i\in\mathbf{i}} r_i \in W(\mathcal{Q}) \tag{2.1.7}$$

is well-defined. Also, since $\mathbf{a}\cdot r_i = r_{\mathbf{a}(i)}\cdot\mathbf{a}$, we see that $s_{\mathbf{i}}$ commutes with the action of $\mathbf{a}$ on $\mathbb{Z}\mathcal{I}$. We denote by $C_{\mathbf{a}}(W(\mathcal{Q}))$ the subgroup of $W(\mathcal{Q})$ consisting of all such elements.

Lemma 2.1.2. *Let $\alpha, \beta \in (\mathbb{Z}\mathcal{I})^{\langle \mathbf{a}\rangle}$. Then*

1. $(\alpha, \beta)_{\mathcal{Q}} = (f(\alpha), f(\beta))_\Gamma$;
2. $f(s_{\mathbf{i}}(\alpha)) = r_{\mathbf{i}}(f(\alpha)) \in \mathbb{Z}\mathbf{I}$;
3. *the map $r_{\mathbf{i}} \mapsto s_{\mathbf{i}}$ induces an isomorphism $W(\Gamma) \xrightarrow{\sim} C_{\mathbf{a}}(W(\mathcal{Q}))$.*

Proof.
Let $i_v \in \mathcal{I}$ enumerate the vertices in the orbit $\mathbf{i} \in \mathbf{I}$. Parts 1 and 2 now follow from the formula

$$b_{\mathbf{ij}} = \sum_{v,w} a_{i_v j_w} = d_{\mathbf{i}} \sum_{w} a_{i_v j_w} \quad \text{for any } v.$$

We denote the length of an element $w \in W(\mathcal{Q})$ by $\ell(w)$. It is shown in [25] that

$$\ell(wr_i) < \ell(w) \quad \text{if and only if} \quad w(e_i) < 0$$

and

$$\ell(w) = \#\{\alpha \in \Delta(\mathcal{Q})_+ \mid w(\alpha) < 0\}.$$

Since $\mathbf{a}$ is admissible and preserves the partial order $\geq$ on $\mathbb{Z}\mathcal{I}$, induction on length shows that $C_{\mathbf{a}}(W(\mathcal{Q}))$ is generated by the $s_{\mathbf{i}}$. Now suppose that $r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_m} = 1 \in W(\Gamma)$. Then $s_{\mathbf{i}_1} \cdots s_{\mathbf{i}_m}$ fixes $(\mathbb{Z}\mathcal{I})^{\langle \mathbf{a} \rangle}$ and so $s_{\mathbf{i}_1} \cdots s_{\mathbf{i}_m}(e_j) > 0$ for all j. Therefore $s_{\mathbf{i}_1} \cdots s_{\mathbf{i}_m} = 1 \in W(\mathcal{Q})$ and the $r_{\mathbf{i}}$ and $s_{\mathbf{i}}$ satisfy the same relations.
■

Proposition 2.1.3. *Let $\alpha \in \Delta(\mathcal{Q})$ and let $r \geq 1$ be minimal such that $\mathbf{a}^r(\alpha) = \alpha$. Set*

$$\sigma(\alpha) := \alpha + \mathbf{a}(\alpha) + \cdots + \mathbf{a}^{r-1}(\alpha) \in (\mathbb{Z}\mathcal{I})^{\langle \mathbf{a} \rangle}.$$

Then $\alpha \mapsto f(\sigma(\alpha))$ induces a surjection $\Delta(\mathcal{Q}) \twoheadrightarrow \Delta(\Gamma)$. Moreover, if $f(\sigma(\alpha))$ is real, then α must also be real and unique up to $\mathbf{a}$-orbit.

Proof.
Set $\beta := f(\sigma(\alpha))$ and consider $w'(\beta)$ for some $w' \in W(\Gamma)$. Let $w \in C_{\mathbf{a}}(\mathcal{Q})$ correspond to w'. Then $w(\sigma(\alpha)) = \sigma(w(\alpha))$ and $w(\alpha) \in \Delta(\mathcal{Q})$.

Since $\mathbf{a}$ preserves the partial order on $\mathbb{Z}\mathcal{I}$, $w'(\beta)$ is either positive or negative. Also, $f(\sigma(w(\alpha)))$ always has connected support. Therefore, if w' is chosen so that $w'(\beta)$ has minimal height, then either $w'(\beta)$ (or its negative) lies in the fundamental region or else it is a multiple of a real root, say $w'(\beta) = me_{\mathbf{i}}$. In the latter case we must have that $w(\alpha) = me_i$ for some vertex i in the $\mathbf{i}$-th orbit, so $w(\alpha) = e_i$ and $m = 1$. This proves that every root of $\mathcal{Q}$ gives rise to a root of Γ.

Conversely, every simple root $e_{\mathbf{i}}$ lies in the image of this map, so consider $\beta \in F_\Gamma$. Then $\gamma = f^{-1}(\beta) \in (\mathbb{Z}\mathcal{I})^{\langle \mathbf{a} \rangle}$ satisfies

$$0 \geq (\beta, e_{\mathbf{i}})_\Gamma = (\gamma, \sigma(e_i))_\mathcal{Q} = \sum_r (\gamma, \mathbf{a}^r(e_i))_\mathcal{Q} = d_{\mathbf{i}}(\gamma, e_i)_\mathcal{Q}$$

for all $i \in \mathbf{i}$. Thus if α is a connected component of γ, then α lies in the fundamental region for $\mathcal{Q}$ and $\sigma(\alpha) = \gamma$.

The result now follows from the correspondence between the Weyl groups.
■

In particular, we note that $\Delta(\mathcal{Q})$ is finite if and only if $\Delta(\Gamma)$ is.

2.2 Automorphisms of Path Algebras

Let $(\mathcal{Q}, \mathbf{a})$ be a quiver with automorphism. It is clear that for any field K, $\mathbf{a}$ induces an algebra automorphism of the path algebra $K\mathcal{Q}$, which we again denote by $\mathbf{a}$. We can also view the path algebra as the tensor algebra $T(\Lambda_0, \Lambda_1)$, where Λ_0 is the semisimple algebra

$$\Lambda_0 := \prod_{i \in \mathcal{I}} K\varepsilon_i \tag{2.2.1}$$

and Λ_1 is the Λ_0-bimodule

$$\Lambda_1 := \prod_{\rho \in \mathcal{A}} K\rho. \tag{2.2.2}$$

We note that $\mathcal{Q}$ has no edge loops precisely when $\varepsilon_i \Lambda_1 \varepsilon_i = 0$ for all vertices i.

It follows that $\mathbf{a}$ is actually a graded automorphism, since by definition it is given by an algebra automorphism of Λ_0 and a Λ_0-bimodule automorphism of Λ_1. Also, $\mathbf{a}$ is admissible if and only if

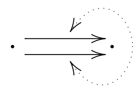
$$\varepsilon_i \Lambda_1 \varepsilon_j = 0 \quad \text{whenever } i \text{ and } j \text{ lie in the same vertex orbit.} \tag{2.2.3}$$

Now suppose that K is an algebraically closed field and n is a positive integer invertible in K. Let $\mathbf{a}$ be a graded automorphism of $K\mathcal{Q}$ of order n satisfying (2.2.3). Then $\mathbf{a}$ necessarily acts as a permutation of the idempotents ε_i and hence induces a permutation of the vertex set $\mathcal{I}$. Also, if $\mathbf{a}^t$ fixes two vertices i and j, then we can find a basis for $\varepsilon_j \Lambda_1 \varepsilon_i$ with respect to which $\mathbf{a}^t$ acts diagonally. In terms of the quiver $\mathcal{Q}$, this means that $\mathbf{a}^t$ acts on each arrow $i \to j$ as multiplication by an n-th root of unity (or more precisely an n/t-th root of unity).

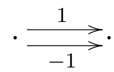
We can again form a symmetrisable GCM $C = D^{-1}B$ using the formulae (2.1.1) and (2.1.2), and hence obtain a valued graph Γ. Furthermore, the results of Lemma 2.1.2 and Proposition 2.1.3 still hold, since their proofs relied only upon the action of $\mathbf{a}$ on the set of vertices $\mathcal{I}$.

From now on, when we say $(\mathcal{Q}, \mathbf{a})$ is a quiver with an admissible automorphism, we shall also include this more general situation.

Example 2.2.1. *Let $\mathcal{Q}$ be the affine quiver $\widetilde{\mathbb{A}}_1$ with the automorphism exchanging the two arrows*



Let K be an algebraically closed field not of characteristic 2. If we call the arrows ρ and $\mathbf{a}(\rho)$, then $\mathbf{a}$ acts diagonally on Λ_1 with respect to the basis given by $\sigma = \frac{1}{2}(\rho + \mathbf{a}(\rho))$ and $\tau = \frac{1}{2}(\rho - \mathbf{a}(\rho))$. We can express this pictorially as



2.3 Isomorphically Invariant Representations

Let $(\mathcal{Q}, \mathbf{a})$ be a quiver with admissible automorphism of order n and K an algebraically closed field of characteristic not dividing n. We shall show how $\mathbf{a}$ determines an autoequivalence of the category of $K\mathcal{Q}$-modules and hence a functor on $\mathrm{Rep}(\mathcal{Q}, K)$. We fix a primitive n-th root of unity ζ in K.

Let $\mathcal{X}$ be a $K\mathcal{Q}$-module. We define a module ${}^{\mathbf{a}}\mathcal{X}$ by taking the same underlying vector space as $\mathcal{X}$ but with new action

$$p \cdot x := \mathbf{a}^{-1}(p)x \quad \text{for } p \in K\mathcal{Q}. \tag{2.3.1}$$

If $\phi : \mathcal{X} \to \mathcal{Y}$ is a module homomorphism, then we obtain a homomorphism ${}^{\mathbf{a}}\phi : {}^{\mathbf{a}}\mathcal{X} \to {}^{\mathbf{a}}\mathcal{Y}$ as follows. Since ϕ is *a fortiori* a vector space map, we set ${}^{\mathbf{a}}\phi = \phi$. Then

$$\phi(p \cdot x) = \phi(\mathbf{a}^{-1}(p)x) = \mathbf{a}^{-1}(p)\phi(x) = p \cdot \phi(x). \tag{2.3.2}$$

In this way we obtain an additive functor $F(\mathbf{a})$ on $\operatorname{mod} K\mathcal{Q}$ such that $F(\mathbf{a}^r) = F(\mathbf{a})^r$. In particular, we note that $\mathcal{X}$ is indecomposable if and only if ${}^{\mathbf{a}}\mathcal{X}$ is.

We know that the categories $\operatorname{mod} K\mathcal{Q}$ and $\operatorname{Rep}(\mathcal{Q}, K)$ are equivalent, so the functor $F(\mathbf{a})$ must act on $\operatorname{Rep}(\mathcal{Q}, K)$. Let $X = (V_i, f_\rho)$ be a K-representation of $\mathcal{Q}$ and $\mathcal{X}$ the corresponding $K\mathcal{Q}$-module (via the functors described in Section 1.3), so $\mathcal{X}$ has underlying vector space $V = \bigoplus_i V_i$. We wish to describe the representation ${}^{\mathbf{a}}X = (W_i, g_\rho)$ corresponding to the module ${}^{\mathbf{a}}\mathcal{X}$ in terms of the original representation X. Clearly

$$W_i = \varepsilon_i \cdot V = \mathbf{a}^{-1}(\varepsilon_i)V = \varepsilon_{\mathbf{a}^{-1}(i)}V = V_{\mathbf{a}^{-1}(i)}. \tag{2.3.3}$$

Now suppose that $\rho : i \to j$ is an arrow. Let $t \geq 1$ be minimal such that $\mathbf{a}^t$ fixes both i and j (so $t = \operatorname{lcm}(d_{\mathbf{i}}, d_{\mathbf{j}})$, where i and j are in the **i**-th and **j**-th orbits respectively). By our choice of basis for Λ_1 (and hence choice of arrows for $\mathcal{Q}$), we have $\mathbf{a}^t(\rho) = \zeta^{tu}\rho$ for some u.

Let $\rho_m := \mathbf{a}^m(\rho)$ for $m = 0, \dots, t-1$. Then by definition,

$$f_{\rho_m}(x) = \rho_m x \quad \text{for all } 0 \leq m < t \tag{2.3.4}$$

and so

$$g_{\rho_m}(x) = \rho_m \cdot x = \begin{cases} \rho_{m-1}x = f_{\rho_{m-1}}(x) & \text{for } 1 \leq m < t; \\ \zeta^{-tu}\rho_{t-1}x = \zeta^{-tu}f_{\rho_{t-1}}(x) & \text{for } m = 0. \end{cases} \tag{2.3.5}$$

We say that a representation X is isomorphically invariant (an ii-representation) if ${}^{\mathbf{a}}X \cong X$. Note that $\underline{\dim}\, {}^{\mathbf{a}}X = \mathbf{a}(\underline{\dim}\, X)$, so any ii-representation has dimension vector fixed by $\mathbf{a}$. We say that X is an ii-indecomposable if it is not isomorphic to the proper direct sum of two ii-representations.

Lemma 2.3.1. *The ii-indecomposables X are precisely the representations of the form*

$$X \cong Y \oplus {}^{\mathbf{a}}Y \oplus \cdots \oplus {}^{\mathbf{a}^{m-1}}Y,$$

where Y is an indecomposable $\mathcal{Q}$-representation and $m \geq 1$ is minimal such that ${}^{\mathbf{a}^m}Y \cong Y$. Moreover, the Krull-Remak-Schmidt Theorem holds for ii-representations.

N.B. The full subcategory with objects the ii-representations is not abelian.

Proof.
Let X be an ii-representation. Viewing X solely as a $\mathcal{Q}$-representation, we can write it as a direct sum of indecomposable representations. Also, since ${}^{\mathbf{a}}X \cong X$, $F(\mathbf{a})$ must act (up to isomorphism) as a permutation of these indecomposable summands. That is, we can write

$$X \cong Z_1 \oplus \cdots \oplus Z_m,$$

where each Z_i is of the form

$$Z_i = Y_i \oplus {}^{\mathbf{a}}Y_i \cdots \oplus {}^{\mathbf{a}^{m_i-1}}Y_i$$

with Y_i indecomposable and $m_i \geq 1$ minimal such that ${}^{\mathbf{a}^{m_i}}Y_i \cong Y_i$.

The lemma follows from the Krull-Remak-Schmidt Theorem for $\mathcal{Q}$-representations.

■

We are now in a position to prove Part 2 and one direction of Part 1 of Theorem 1.6.1.

Proposition 2.3.2. *Let $(\mathcal{Q}, \mathbf{a})$ be a quiver with an admissible automorphism; Γ the associated valued graph and K an algebraically closed field of characteristic not dividing the order of $\mathbf{a}$. Then there is an ii-indecomposable of dimension vector $f^{-1}(\alpha)$ only if α is a root of Γ.*

Moreover, every positive real root of Γ occurs and the corresponding ii-indecomposable is unique up to isomorphism with $\frac{1}{2}(\alpha,\alpha)_\Gamma$ indecomposable summands.

Proof.
Let Y be an indecomposable representation of $\mathcal{Q}$ and consider the ii-indecomposable

$$X \cong Y \oplus {}^{\mathbf{a}}Y \oplus \cdots \oplus {}^{\mathbf{a}^{m-1}}Y.$$

Writing $\beta = \underline{\dim}\, Y \in \Delta(\mathcal{Q})_+$, then $\underline{\dim}\, X = \beta + \mathbf{a}(\beta) + \cdots + \mathbf{a}^{m-1}(\beta) = r\sigma(\beta)$ for some r. If β is real, then by Kac's Theorem $Y \cong {}^{\mathbf{a}^m}Y$ if and only if $\mathbf{a}^m(\beta) = \beta$

and so $r = 1$. Therefore $f(\underline{\dim}\, X) = f(\sigma(\beta)) \in \Delta(\Gamma)_+$. On the other hand, if β is imaginary, then so is $f(\sigma(\beta))$ and hence $f(\underline{\dim}\, X) = rf(\sigma(\beta)) \in \Delta(\Gamma)_+$.

This shows that the dimension vectors of ii-indecomposables give rise to positive roots of Γ. Also, it follows from Proposition 2.1.3 and Kac's Theorem that we must get every real root of Γ and that the corresponding ii-indecomposable is unique up to isomorphism with the stated number of indecomposable summands. ■

Of special interest in the representation theory of quivers (over an arbitrary field K) are the reflection functors R_i^+ and R_i^-, defined when $i \in \mathcal{I}$ is a sink or a source respectively (see for example [8]). Clearly if i is a sink (respectively a source) then the same is true for all vertices in the orbit of i. Also, since $\mathbf{a}$ is admissible, the functor

$$S_{\mathbf{i}}^{\pm} := \prod_{i \in \mathbf{i}} R_i^{\pm}$$

is well-defined.

Denote by K_i the i-th simple representation of $\mathcal{Q}$. If $d = d_{\mathbf{i}}$ is the size of the orbit of i, then we have an ii-indecomposable

$$K_{\mathbf{i}} := K_i \oplus K_{\mathbf{a}(i)} \oplus \cdots K_{\mathbf{a}^{d-1}(i)}.$$

Proposition 2.3.3. *Let i be a sink (or a source, interchanging $+$ and $-$). Then for any ii-representation X there is a canonical monomorphism*

$$\phi_X : S_{\mathbf{i}}^- S_{\mathbf{i}}^+(X) \to X$$

whose image has a complement a direct sum of copies of the ii-indecomposable $K_{\mathbf{i}}$. In fact,

1. $S_{\mathbf{i}}^+(K_{\mathbf{i}}) = 0$;

2. *if $X \neq K_{\mathbf{i}}$ is an ii-indecomposable, then ϕ_X is an isomorphism and hence*

$$\operatorname{End}(S_{\mathbf{i}}^+(X)) \cong \operatorname{End}(X) \qquad \text{and} \qquad \underline{\dim}\, S_{\mathbf{i}}^+(X) = s_{\mathbf{i}}(\underline{\dim}\, X).$$

Proof.
This follows immediately from the standard properties of reflection functors [8].
■

2.4 Geometrical Aspects

Let K be an algebraically closed field. As in Section 1.6, we can consider the subset Rep-ii(α, K) of Rep$(f^{-1}(\alpha), K)$ corresponding to the set of all ii-representations of dimension vector $f^{-1}(\alpha)$, where $0 \leq \alpha \in \mathbb{Z}\mathbf{I}$. This is clearly GL$(f^{-1}(\alpha), K)$-stable and is also a constructible subset. For, consider the affine variety

$$M(\alpha, K) := \{(g, X) \in \mathrm{GL}(f^{-1}(\alpha), K) \times \mathrm{Rep}(f^{-1}(\alpha), K) \mid g \cdot X = {}^{\mathbf{a}}X\}. \quad (2.4.1)$$

Then the image of $M(\alpha, K)$ under the projection to the second co-ordinate is precisely the set Rep-ii(α, K), which is constructible by Chevalley's Theorem.

Theorem 2.4.1 (Chevalley). *Let $f : Y \to X$ be a morphism of varieties and let $Z \subset Y$ be a constructible subset. Then the image $f(Z)$ is constructible in X. In particular, $f(Y)$ is constructible.*

Similarly for any proper decomposition $\alpha = \beta + \gamma$ we can consider the morphism

$$\theta_{\beta,\gamma} : \mathrm{GL}(f^{-1}(\alpha), K) \times \mathrm{Rep}(f^{-1}(\beta), K) \times \mathrm{Rep}(f^{-1}(\gamma), K) \to \mathrm{Rep}(f^{-1}(\alpha), K),$$
$$(g, X, Y) \mapsto g \cdot (X \oplus Y). \quad (2.4.2)$$

The image under $\theta_{\beta,\gamma}$ of the constructible subset GL$(f^{-1}(\alpha), K) \times$ Rep-ii$(\beta, K) \times$ Rep-ii(γ, K) is contained in Rep-ii(α, K) and the complement in Rep-ii(α, K) of the union of all such images is precisely the set Ind-ii(α, K), corresponding to all ii-indecomposables of dimension vector $f^{-1}(\alpha)$. Thus this set is also GL$(f^{-1}(\alpha), K)$-stable and constructible.

Therefore we can define the number of parameters

$$\mu\text{-}ii(\alpha, K) := \dim_{\mathrm{GL}(f^{-1}(\alpha),K)} \text{Ind-ii}(\alpha, K) \quad (2.4.3)$$

and the number of top-dimensional families of orbits

$$t\text{-}ii(\alpha, K) := \mathrm{top}_{\mathrm{GL}(f^{-1}(\alpha),K)} \, \mathrm{Ind\text{-}ii}(\alpha, K) \tag{2.4.4}$$

for the action of $\mathrm{GL}(f^{-1}(\alpha), K)$ on $\mathrm{Ind\text{-}ii}(\alpha, K)$.

In fact, in Chapter 5 we take this idea further and construct affine schemes $\mathrm{Rep}(f^{-1}(\alpha))$, $\mathrm{GL}(f^{-1}(\alpha))$ and $M(\alpha)$ over the integers. We then show that for K an algebraically closed field, the sets $\mathrm{Rep\text{-}ii}(\alpha, K)$ and $\mathrm{Ind\text{-}ii}(\alpha, K)$ can be thought of as the K-rational points of constructible subsets of the affine scheme $\mathrm{Rep}(f^{-1}(\alpha))$.

Chapter 3

Skew Group Algebras

We fix a quiver with admissible automorphism $(\mathcal{Q}, \mathbf{a})$ and write n for the order of $\mathbf{a}$. Let K be an algebraically closed field of characteristic not dividing n. We fix a primitive n-th root of unity $\zeta \in K$.

In this chapter we first show, using an argument of Gabriel [12], that every ii-module for $K\mathcal{Q}$ can be thought of as the restriction of a module for the skew group algebra $K\mathcal{Q}\#\langle\mathbf{a}\rangle$. This latter algebra is Morita equivalent to another path algebra $K\widetilde{\mathcal{Q}}$ on which the group of characters of $\langle\mathbf{a}\rangle$, say generated by $\tilde{\mathbf{a}}$, acts admissibly. The pair $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ is called the dual quiver with automorphism. We obtain Morita equivalences

$$\operatorname{mod} K\mathcal{Q}\#\langle\mathbf{a}\rangle \sim \operatorname{mod} K\widetilde{\mathcal{Q}} \qquad \text{and} \qquad \operatorname{mod} K\widetilde{\mathcal{Q}}\#\langle\tilde{\mathbf{a}}\rangle \sim \operatorname{mod} K\mathcal{Q}.$$

We naturally have induction and restriction functors between $\operatorname{mod} K\mathcal{Q}$ and $\operatorname{mod} K\mathcal{Q}\#\langle\mathbf{a}\rangle$. These are left and right adjoints of each other and so combining these with the Morita equivalences gives left and right adjoint functors between $\operatorname{mod} K\mathcal{Q}$ and $\operatorname{mod} K\widetilde{\mathcal{Q}}$. On the other hand, we could have started with $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ and obtained a different pair of functors. We show that there are natural isomorphisms between these two pairs of functors.

Using this, we show that the dimension vectors of the ii-indecomposables are precisely the positive roots of Γ, where Γ is the valued graph associated to the pair $(\mathcal{Q}, \mathbf{a})$. This completes the proof of Theorem 1.6.1.

3.1 Automorphisms of ii-Representations

Proposition 3.1.1. *Let X be an ii-representation. Then there exists an isomorphism $\phi : {}^{\mathbf{a}}X \to X$ such that the automorphism $\phi\, {}^{\mathbf{a}}\phi \cdots {}^{\mathbf{a}^{n-1}}\phi$ of X is the identity.*

The following proof is taken from [12], Proposition 3.9.

Proof.
Let X be an ii-representation and $\theta : {}^{\mathbf{a}}X \to X$ an isomorphism. Write

$$\Theta := \theta\, {}^{\mathbf{a}}\theta \cdots {}^{\mathbf{a}^{n-1}}\theta \in \mathrm{Aut}(X),$$

where ${}^{\mathbf{a}}\theta$ is the corresponding map ${}^{\mathbf{a}^2}X \to {}^{\mathbf{a}}X$. We note that ${}^{\mathbf{a}}\Theta = \theta^{-1}\Theta\theta$ and so ${}^{\mathbf{a}}\psi = \theta^{-1}\psi\theta$ for all $\psi \in K[\Theta]$.

Pick $\psi \in K[\Theta]$ and set $\phi := \psi\theta$. By induction,

$$\phi\, {}^{\mathbf{a}}\phi \cdots {}^{\mathbf{a}^{m-1}}\phi = \psi^m\, \theta\, {}^{\mathbf{a}}\theta \cdots {}^{\mathbf{a}^{m-1}}\theta,$$

so in particular,

$$\phi\, {}^{\mathbf{a}}\phi \cdots {}^{\mathbf{a}^{n-1}}\phi = \psi^n\, \Theta.$$

Since K is algebraically closed, the characteristic equation for Θ is a product of linear factors. Therefore $K[\Theta]$ is isomorphic to a product of rings of the form $K[T]/(T-\lambda)^m$ for $\lambda \neq 0$.

Let $x = x_0 + x_1(T-\lambda) + \cdots + x_{m-1}(T-\lambda)^{m-1}$ be an element of $K[T]/(T-\lambda)^m$. We can find an n-th root of x in $K[T]/(T-\lambda)^m$ whenever $x_0 \neq 0$. This can be seen by successively working modulo $(T-\lambda)^r$ for $r = 1, \ldots, m$, using that n is invertible in K. (In fact, the n-th root of x is determined by the choice of n-th root of x_0.)

Since Θ^{-1} is congruent to λ^{-1} in $K[T]/(T-\lambda)$, we can find an element $\psi \in K[\Theta]$ such that $\psi^n\Theta = 1$. Thus $\phi = \psi\theta$ is the required isomorphism.
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Below we shall see that the pair (X, ϕ) corresponds to a module for the skew group algebra $K\mathcal{Q}\#\langle\mathbf{a}\rangle$.

3.2 Skew Group Algebras

We know that $\mathbf{a}$ determines an automorphism of the path algebra $K\mathcal{Q}$ and so we can form the skew group algebra $K\mathcal{Q}\#\langle\mathbf{a}\rangle$. This has K-basis the elements $p\mathbf{a}^r$, with p a path in $K\mathcal{Q}$, and multiplication

$$p\mathbf{a}^r \cdot q\mathbf{a}^s := p\mathbf{a}^r(q)\mathbf{a}^{r+s}. \tag{3.2.1}$$

Suppose that $\mathcal{X}$ is a $K\mathcal{Q}\#\langle\mathbf{a}\rangle$-module. Then $\mathcal{X}$ is clearly a $K\mathcal{Q}$-module by restriction and $\mathbf{a}$ acts on $\mathcal{X}$ as a linear map such that $\mathbf{a}^n = 1$. Also,

$$\mathbf{a}(px) = (\mathbf{a}p)x = (\mathbf{a}(p)\mathbf{a})x = \mathbf{a}(p)(\mathbf{a}x). \tag{3.2.2}$$

That is, the map $x \mapsto \mathbf{a}x$ gives a $K\mathcal{Q}$-module isomorphism $\phi : {}^{\mathbf{a}}\mathcal{X} \to \mathcal{X}$ such that

$$\phi\, {}^{\mathbf{a}}\phi \cdots {}^{\mathbf{a}^{n-1}}\phi = 1. \tag{3.2.3}$$

Conversely, any pair $(\mathcal{X}, \phi)$ consisting of a $K\mathcal{Q}$-module $\mathcal{X}$ and a module isomorphism $\phi : {}^{\mathbf{a}}\mathcal{X} \to \mathcal{X}$ satisfying (3.2.3) can be given the structure of a $K\mathcal{Q}\#\langle\mathbf{a}\rangle$-module. In particular, given any $K\mathcal{Q}$-module $\mathcal{X}$ such that ${}^{\mathbf{a}}\mathcal{X} \cong \mathcal{X}$, we can find ϕ such that $(\mathcal{X}, \phi)$ is a $K\mathcal{Q}\#\langle\mathbf{a}\rangle$-module. We note, however, that there may exist two such maps ϕ and ϕ' with $(\mathcal{X}, \phi)$ and $(\mathcal{X}, \phi')$ non-isomorphic.

3.3 The Dual Quiver

It is shown in [36] that if $\mathcal{Q}$ has no oriented cycles (so $K\mathcal{Q}$ is an artin algebra) and $\mathbf{a}$ is an admissible automorphism of order n invertible in K, then the skew group algebra $K\mathcal{Q}\#\langle\mathbf{a}\rangle$ is Morita equivalent to the path algebra of another quiver.

Moreover, since $\mathbf{a}$ acts as a graded automorphism of the tensor algebra $T(\Lambda_0, \Lambda_1)$, we have that

$$T(\Lambda_0, \Lambda_1)\#\langle\mathbf{a}\rangle \cong T(\Lambda_0\#\langle\mathbf{a}\rangle, \Lambda_1\#\langle\mathbf{a}\rangle). \tag{3.3.1}$$

Here $\Lambda_0\#\langle\mathbf{a}\rangle$ is the skew group algebra and $\Lambda_1\#\langle\mathbf{a}\rangle$ is the $\Lambda_0\#\langle\mathbf{a}\rangle$-bimodule with K-basis the elements $\rho\mathbf{a}^m$ for $\rho \in \mathcal{A}$ and $0 \le m < n$.

It now follows from the construction of [36], which we outline below, that even when $\mathcal{Q}$ has oriented cycles, the skew-group algebra $K\mathcal{Q}\#\langle\mathbf{a}\rangle$ is always Morita equivalent to the path algebra of another quiver. We shall call this quiver the dual quiver and denote it by $\widetilde{\mathcal{Q}}$.

Consider first the algebra $\Lambda_0\#\langle\mathbf{a}\rangle$. We can label the vertices in $\mathcal{I}$ as pairs $(\mathbf{i},r)$ for $\mathbf{i}\in\mathbf{I}$ and $r\in\mathbb{Z}/d_{\mathbf{i}}\mathbb{Z}$ such that $\mathbf{a}(\mathbf{i},r)=(\mathbf{i},r+1)$. Now pick $\mathbf{i}\in\mathbf{I}$ and let $d=d_{\mathbf{i}}$. Set

$$R_{\mathbf{i}}:=K\varepsilon_{(\mathbf{i},0)}\times K\varepsilon_{(\mathbf{i},1)}\times\cdots\times K\varepsilon_{(\mathbf{i},d-1)}. \tag{3.3.2}$$

We also introduce the elements

$$z(d,\mu):=\frac{d}{n}\sum_{m=0}^{(n/d)-1}\zeta^{dm\mu}\mathbf{a}^{dm}\in K\langle\mathbf{a}\rangle \tag{3.3.3}$$

for $0\le\mu<n/d$. These are orthogonal idempotents summing to 1 in the group algebra $K\langle\mathbf{a}\rangle$ and which centralise $R_{\mathbf{i}}$.

We have an isomorphism

$$\prod_{\mu=0}^{(n/d)-1}\mathbb{M}\left(d,K\right)_{(\mathbf{i},\mu)}\xrightarrow{\sim}R_{\mathbf{i}}\#\langle\mathbf{a}\rangle,\qquad E(\mathbf{i},\mu)_{pq}\mapsto\varepsilon_{(\mathbf{i},p)}\mathbf{a}^{p-q}z(d,\mu), \tag{3.3.4}$$

where $\mathbb{M}\left(d,K\right)$ is the algebra of $d\times d$ matrices over K and the $E(\mathbf{i},\mu)_{pq}$ for $0\le p,q<d$ are the elementary matrices for the $(\mathbf{i},\mu)$-th copy of $\mathbb{M}\left(d,K\right)$.

If we write

$$\tilde{\varepsilon}_{(\mathbf{i},\mu)}:=E(\mathbf{i},\mu)_{00} \tag{3.3.5}$$

then the algebra $R_{\mathbf{i}}\#\langle\mathbf{a}\rangle$ is clearly Morita equivalent to

$$\widetilde{R}_{\mathbf{i}}:=K\tilde{\varepsilon}_{(\mathbf{i},0)}\times K\tilde{\varepsilon}_{(\mathbf{i},1)}\times\cdots\times K\tilde{\varepsilon}_{(\mathbf{i},(n/d)-1)}. \tag{3.3.6}$$

Doing this for each vertex orbit $\mathbf{i}\in\mathbf{I}$, we observe that $\Lambda_0\#\langle\mathbf{a}\rangle$ is Morita equivalent to $\prod_{\mathbf{i}}\widetilde{R}_{\mathbf{i}}$ via the idempotent

$$E:=\sum_{\mathbf{i}\in\mathbf{I}}\sum_{\mu=0}^{(n/d_{\mathbf{i}})-1}E(\mathbf{i},\mu)_{00}. \tag{3.3.7}$$

In particular, the vertices $\widetilde{\mathcal{I}}$ of $\widetilde{\mathcal{Q}}$ can be considered as pairs $(\mathbf{i},\mu)$ with $\mathbf{i}\in\mathbf{I}$ and $0\le\mu<n/d_{\mathbf{i}}$.

We now determine the arrows of $\widetilde{\mathcal{Q}}$. Since the algebra $\Lambda_0\#\langle\mathbf{a}\rangle$ is semisimple, the bimodule $\Lambda_1\#\langle\mathbf{a}\rangle$ must decompose as a direct sum of simple modules. Furthermore, each of these will be generated by an arrow. We simplify our discussion by considering each simple module separately.

Let $\rho:(\mathbf{i},0)\to(\mathbf{j},l)$ be an arrow of $\mathcal{Q}$ and set $t:=\operatorname{lcm}(d_{\mathbf{i}},d_{\mathbf{j}})$, where $d_{\mathbf{i}}$ and $d_{\mathbf{j}}$ are the sizes of the $\mathbf{i}$-th and $\mathbf{j}$-th orbits respectively. Then $\mathbf{a}^t$ fixes both $(\mathbf{i},0)$ and $(\mathbf{j},l)$ and so $\mathbf{a}^t(\rho)=\zeta^{tu}\rho$ for some u. If M is the bimodule generated by ρ, then EME is a bimodule for $\prod_{\mathbf{i}}\widetilde{R}_{\mathbf{i}}$ generated by the $E(\mathbf{j},\nu)_{0l}\rho E(\mathbf{i},\mu)_{00}$, for various μ and ν.

Expanding, we have that

$$E(\mathbf{j},\nu)_{0l}\rho E(\mathbf{i},\mu)_{00}=\frac{d_{\mathbf{i}}d_{\mathbf{j}}}{n^2}\sum_{p=0}^{(n/d_{\mathbf{i}})-1}\sum_{q=0}^{(n/d_{\mathbf{j}})-1}\zeta^{d_{\mathbf{i}}p\mu+d_{\mathbf{j}}q\nu}\mathbf{a}^{d_{\mathbf{j}}q-l}(\rho)\mathbf{a}^{d_{\mathbf{i}}p+d_{\mathbf{j}}q-l}.$$

If we write $d_{\mathbf{i}}p=Pt+d_{\mathbf{i}}p'$ with $0\le P<n/t$ and $0\le p'<t/d_{\mathbf{i}}$, and similarly for $d_{\mathbf{j}}q=Qt+d_{\mathbf{j}}q'$, then this becomes

$$\frac{d_{\mathbf{i}}d_{\mathbf{j}}}{n^2}\sum_{P,Q,p',q'}\zeta^{(P+Q)t\mu+Qt(u+\nu-\mu)+d_{\mathbf{i}}p'\mu+d_{\mathbf{j}}q'\nu}\mathbf{a}^{d_{\mathbf{j}}q'-l}(\rho)\mathbf{a}^{(P+Q)t+d_{\mathbf{i}}p'+d_{\mathbf{j}}q'-l}.$$

Finally, if we write $d_{\mathbf{i}}r\equiv(P+Q)t+d_{\mathbf{i}}p' \bmod n$ with $0\le r<n/d_{\mathbf{i}}$, then we obtain the factorisation

$$\Big(\frac{t}{n}\sum_{Q}\zeta^{Qt(u+\nu-\mu)}\Big)\Big(\frac{d_{\mathbf{i}}d_{\mathbf{j}}}{nt}\sum_{r,q'}\zeta^{d_{\mathbf{i}}r\mu+d_{\mathbf{j}}q'\nu}\mathbf{a}^{d_{\mathbf{j}}q'-l}(\rho)\mathbf{a}^{d_{\mathbf{i}}r+d_{\mathbf{j}}q'-l}\Big).$$

Since the elements $\mathbf{a}^{d_{\mathbf{j}}q'-l}(\rho)\mathbf{a}^{d_{\mathbf{i}}r+d_{\mathbf{j}}q'-l}$ are linearly independent, we see that $E(\mathbf{j},\nu)_{0l}\rho E(\mathbf{i},\mu)_{00}$ is non-zero if and only if $\mu\equiv u+\nu \bmod n/t$, in which case it equals

$$\Big(\frac{d_{\mathbf{j}}}{t}\sum_{q'}\zeta^{d_{\mathbf{j}}q'\nu}\mathbf{a}^{d_{\mathbf{j}}q'-l}(\rho)\mathbf{a}^{d_{\mathbf{j}}q'-l}\Big)z(d_{\mathbf{i}},\mu). \tag{3.3.8}$$

In conclusion, from the simple module corresponding to the arrow $\rho:(\mathbf{i},0)\to(\mathbf{j},l)$ of $\mathcal{Q}$, we get arrows $(\mathbf{i},\mu)\to(\mathbf{j},\nu)$ in $\widetilde{\mathcal{Q}}$ for $\mu\equiv u+\nu \bmod n/t$.

3.4 The Dual Group Action

We keep the notation of the last section. Let $\tilde{\mathbf{a}}$ be a generator for the group of characters of $\langle \mathbf{a} \rangle$ such that $\tilde{\mathbf{a}}(\mathbf{a}) = \zeta$. It is proved in [36] that $\tilde{\mathbf{a}}$ acts naturally on the skew group algebra $K\mathcal{Q}\#\langle \mathbf{a} \rangle$ via

$$\tilde{\mathbf{a}}(p\mathbf{a}^m) = \tilde{\mathbf{a}}(\mathbf{a}^m)p\mathbf{a}^m = \zeta^m p\mathbf{a}^m \tag{3.4.1}$$

and that the algebra $(K\mathcal{Q}\#\langle \mathbf{a} \rangle)\#\langle \tilde{\mathbf{a}} \rangle$ is Morita equivalent to $K\mathcal{Q}$ (see Proposition 3.5.2).

Let us consider how $\tilde{\mathbf{a}}$ acts on the idempotents $E(\mathbf{i}\mu)_{00}$. We have

$$\tilde{\mathbf{a}}(z(d,\mu)) = \tilde{\mathbf{a}}\Big(\frac{d}{n}\sum_m \zeta^{dm\mu}\mathbf{a}^{dm}\Big) = \frac{d}{n}\sum_m \zeta^{dm(\mu+1)}\mathbf{a}^{dm} = z(d,\mu+1) \tag{3.4.2}$$

and hence

$$\tilde{\mathbf{a}}(E(\mathbf{i},\mu)_{00}) = E(\mathbf{i},\mu+1)_{00}. \tag{3.4.3}$$

In particular, the idempotent E is fixed by $\tilde{\mathbf{a}}$ and so we have an induced action of $\tilde{\mathbf{a}}$ on $K\widetilde{\mathcal{Q}}$. This sends vertex $(\mathbf{i},\mu)$ to the vertex $(\mathbf{i},\mu+1)$ and thus is again admissible. (C.f. the action of $\mathbf{a}$ on the vertices $\mathcal{I}$ sending $(\mathbf{i},r)$ to $(\mathbf{i},r+1)$.)

Let $\rho : (\mathbf{i},0) \to (\mathbf{j},l)$ be an arrow of $\mathcal{Q}$, $t = \mathrm{lcm}(d_{\mathbf{i}},d_{\mathbf{j}})$ and $\mathbf{a}^t(\rho) = \zeta^{tu}\rho$. Then we get arrows of $\widetilde{\mathcal{Q}}$ corresponding to the elements $E(\mathbf{j},\nu)_{0l}\rho E(\mathbf{i},\mu)_{00}$ with $\mu \equiv u+\nu \bmod n/t$. Since

$$\tilde{\mathbf{a}}(E(\mathbf{j},\nu)_{0l}) = \zeta^{-l}E(\mathbf{j},\nu+1)_{0l},$$

we get that

$$\tilde{\mathbf{a}}\big(E(\mathbf{j},\nu)_{0l}\rho E(\mathbf{i},\mu)_{00}\big) = \zeta^{-l}E(\mathbf{j},\nu+1)_{0l}\rho E(\mathbf{i},\mu+1)_{00}.$$

Now, $\mathrm{lcm}(\frac{n}{d_{\mathbf{i}}},\frac{n}{d_{\mathbf{j}}}) = \frac{nt}{d_{\mathbf{i}}d_{\mathbf{j}}} = T$ and so $\tilde{\mathbf{a}}^T$ fixes both the vertices $(\mathbf{i},\mu)$ and $(\mathbf{j},\nu)$ of $\widetilde{\mathcal{Q}}$. Then

$$\tilde{\mathbf{a}}^T\big(E(\mathbf{j},\nu)_{0l}\rho E(\mathbf{i},\mu)_{00}\big) = \zeta^{-lT}E(\mathbf{j},\nu)_{0l}\rho E(\mathbf{i},\mu)_{00}. \tag{3.4.4}$$

It follows that if we apply the above construction to $(\widetilde{\mathcal{Q}},\tilde{\mathbf{a}})$, then we recover the original pair $(\mathcal{Q},\mathbf{a})$.

As $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ is again a quiver with admissible automorphism, we may construct the associated valued graph $\widetilde{\Gamma}$. Let $\widetilde{\mathcal{Q}}$ and $\widetilde{\Gamma}$ have generalised Cartan matrices $\widetilde{A}$ and $\widetilde{C} = \widetilde{D}^{-1}\widetilde{B}$ respectively. Clearly the vertex orbits of $\tilde{\mathbf{a}}$ on $\widetilde{\mathcal{Q}}$ are indexed by $\mathbf{I}$ and

$$\tilde{d}_{\mathbf{i}} = n/d_{\mathbf{i}}.$$

Consider a simple bimodule generated by an arrow between the $\mathbf{i}$-th and $\mathbf{j}$-th vertex orbits in $\mathcal{Q}$. Each such module corresponds to $t = \mathrm{lcm}(d_{\mathbf{i}}, d_{\mathbf{j}})$ arrows in $\mathcal{Q}$ and to $nt/d_{\mathbf{i}}d_{\mathbf{j}} = \mathrm{lcm}(\tilde{d}_{\mathbf{i}}, \tilde{d}_{\mathbf{j}})$ arrows in $\widetilde{\mathcal{Q}}$. Therefore

$$\frac{1}{t}\sum_{r,s} a_{(\mathbf{i},r)(\mathbf{j},s)} = \frac{d_{\mathbf{i}}d_{\mathbf{j}}}{nt}\sum_{\mu,\nu} \tilde{a}_{(\mathbf{i},\mu)(\mathbf{j},\nu)}.$$

Thus

$$\widetilde{D} = nD^{-1} \qquad \text{and} \qquad \widetilde{B} = nD^{-1}BD^{-1}, \tag{3.4.5}$$

which implies that

$$\widetilde{C} = BD^{-1} = C^{\mathrm{tr}}. \tag{3.4.6}$$

Therefore Γ and $\widetilde{\Gamma}$ are dual valued graphs, in the sense of [25]. This explains our terminology that $(\mathcal{Q}, \mathbf{a})$ and $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ are dual.

Summing up, let $(\mathcal{Q}, \mathbf{a})$ be a quiver with admissible automorphism (of order n) and with associated valued graph Γ. We can write the vertices $\mathcal{I}$ of $\mathcal{Q}$ as pairs $(\mathbf{i}, r)$, where $\mathbf{i} \in \mathbf{I}$ is a vertex orbit of size $d_{\mathbf{i}}$, $r \in \mathbb{Z}/d_{\mathbf{i}}\mathbb{Z}$ and $\mathbf{a}(\mathbf{i}, r) = (\mathbf{i}, r+1)$.

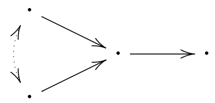
We can also form the dual quiver with admissible automorphism $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ whose associated valued graph $\widetilde{\Gamma}$ is dual to Γ. The vertices $\widetilde{\mathcal{I}}$ of $\widetilde{\mathcal{Q}}$ can be expressed as pairs $(\mathbf{i}, \mu)$, where $\mathbf{i} \in \mathbf{I}$, $\mu \in \mathbb{Z}/(n/d_{\mathbf{i}})\mathbb{Z}$ and $\tilde{\mathbf{a}}(\mathbf{i}, \mu) = (\mathbf{i}, \mu+1)$.

For an algebraically closed field K of characteristic not dividing n, we then have Morita equivalences

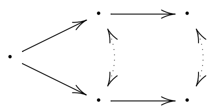
$$K\mathcal{Q}\#\langle \mathbf{a}\rangle \sim K\widetilde{\mathcal{Q}} \qquad \text{and} \qquad K\widetilde{\mathcal{Q}}\#\langle \tilde{\mathbf{a}}\rangle \sim K\mathcal{Q}. \tag{3.4.7}$$

We illustrate these constructions with a couple of examples.

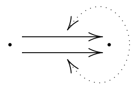
Example 3.4.1. *Consider the quiver of type* $\mathbb{D}_4$ *with automorphism*



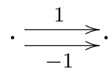
Then the dual quiver is of type $\mathbb{A}_5$ *with automorphism*



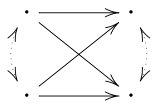
Example 3.4.2. *Next let* $\mathcal{Q}$ *be the affine quiver* $\widetilde{\mathbb{A}}_1$ *with the automorphism exchanging the two arrows*



As in Example 2.2.1, we can change basis to get the action



The dual quiver $\widetilde{\mathcal{Q}}$ *is then of type* $\widetilde{\mathbb{A}}_3$ *with automorphism*



In fact in this example, $K\mathcal{Q}\#\langle \mathbf{a} \rangle \cong K\widetilde{\mathcal{Q}}$ *and* $K\widetilde{\mathcal{Q}}\#\langle \tilde{\mathbf{a}} \rangle \cong \mathbb{M}_2(K\mathcal{Q})$.

3.5 Restriction and Induction Functors

We now consider a slightly more general situation, following [36]. Let K be an algebraically closed base field and let Λ be either an artin algebra or a path algebra over K. Let G be a finite cyclic group whose order is invertible in K and fix a group homomorphism from G to either the group of all automorphisms of Λ in the first case or graded automorphisms in the latter. We write ΛG for the skew group algebra.

We can regard ΛG as a ΛG-Λ-bimodule and so form the induction and restriction functors

$$\begin{aligned} F &:= \Lambda G \otimes_\Lambda - \ : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda G \\ H &:= \text{restriction} : \operatorname{mod}\Lambda G \to \operatorname{mod}\Lambda. \end{aligned} \tag{3.5.1}$$

These functors are both left and right adjoints of each other ([36], Theorem 1.1).

We have ([36], Proposition 1.8)

Proposition 3.5.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be indecomposable Λ-modules. Then*

1. $HF(\mathcal{X}) \cong \bigoplus_{g\in G} {}^g\mathcal{X}$;
2. $F(\mathcal{X}) \cong F(\mathcal{Y})$ *if and only if* $\mathcal{Y} \cong {}^g\mathcal{X}$ *for some* $g \in G$;
3. $F(\mathcal{X})$ *has exactly* m *indecomposable summands, where* m *is the order of the set* $\{g \in G \mid {}^g\mathcal{X} \cong \mathcal{X}\}$.

The dual group $\widehat{G}$ acts on ΛG via $\chi(\lambda g) = \chi(g)\lambda g$. Then ([36], Proposition 5.1)

Proposition 3.5.2. *Consider ΛG as a right Λ-module. Then the map ϕ : $(\Lambda G)\widehat{G} \to \operatorname{End}_\Lambda(\Lambda G)$ given by*

$$\phi(\lambda g\chi) : \mu h \mapsto \chi(h)\lambda g \cdot \mu h = \chi(h)\lambda g(\mu)gh$$

is an algebra isomorphism.

In particular, since ΛG_Λ is a finitely generated projective generator, the algebra $(\Lambda G)\widehat{G}$ is Morita equivalent to Λ.

We use ϕ to make ΛG a $(\Lambda G)\widehat{G}$-Λ-bimodule.

We also consider the functors

$$\begin{aligned} F' &:= (\Lambda G)\widehat{G} \otimes_{\Lambda G} - : \operatorname{mod}\Lambda G \to \operatorname{mod}(\Lambda G)\widehat{G} \\ H' &:= \text{restriction} \qquad : \operatorname{mod}(\Lambda G)\widehat{G} \to \operatorname{mod}\Lambda G, \end{aligned} \tag{3.5.2}$$

and note that the Morita equivalence $\operatorname{mod}\Lambda \to \operatorname{mod}(\Lambda G)\widehat{G}$ is given by

$$M := {}_{(\Lambda G)\widehat{G}}\Lambda G \otimes_\Lambda -. \tag{3.5.3}$$

Corollary 3.5.3. *There are natural isomorphisms*

$$F \cong H'M \qquad \text{and} \qquad F' \cong MH.$$

Proof.

We have $H'M = {}_{\Lambda G}\Lambda G \otimes_\Lambda - = F$. The second isomorphism follows by taking adjoints.

■

It follows that Proposition 3.5.1 holds with F and H interchanged. Namely

Proposition 3.5.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be indecomposable ΛG-modules. Then*

1. $FH(\mathcal{X}) \cong \bigoplus_{\chi \in \widehat{G}} {}^{\chi}\mathcal{X}$;
2. $H(\mathcal{X}) \cong H(\mathcal{Y})$ *if and only if* $\mathcal{Y} \cong {}^{\chi}\mathcal{X}$ *for some* $\chi \in \widehat{G}$;
3. $H(\mathcal{X})$ *has exactly m indecomposable summands, where m is the order of the set* $\{\chi \in \widehat{G} \mid {}^{\chi}\mathcal{X} \cong \mathcal{X}\}$.

Lemma 3.5.5. *Let e be an idempotent of Λ fixed by G and such that $e\Lambda e$ is Morita equivalent to Λ. Then we have natural isomorphisms between*

$$\operatorname{mod}\Lambda \xrightarrow{\Lambda G \otimes_\Lambda -} \operatorname{mod}\Lambda G \xrightarrow{e\Lambda G \otimes_{\Lambda G} -} \operatorname{mod} e\Lambda G e$$

$$\text{and} \qquad \operatorname{mod}\Lambda \xrightarrow{e\Lambda \otimes_\Lambda -} \operatorname{mod} e\Lambda e \xrightarrow{e\Lambda G e \otimes_{e\Lambda e} -} \operatorname{mod} e\Lambda G e\,.$$

Proof.

The only thing to check is that $e\Lambda G e \otimes_{e\Lambda e} e\Lambda$ and $e\Lambda G$ are isomorphic as $e\Lambda G e$-Λ-bimodules. This can be seen via the multiplication map.

■

We now relate this back to quivers with automorphisms.

Let $(\mathcal{Q}, \mathbf{a})$ be a quiver with admissible automorphism of order n and let K be an algebraically closed field of characteristic not dividing n. Set $\Lambda = K\mathcal{Q}$ and $G = \langle \mathbf{a} \rangle$ and construct the functors F, H, F', H' and M as above.

Let $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ be the dual quiver with automorphism, given via the idempotent E in $K\mathcal{Q}\#\langle\mathbf{a}\rangle$, and recall that this idempotent is fixed by $\tilde{\mathbf{a}}$. Let us write $\widetilde{F}$ and $\widetilde{H}$ for the induction and restriction functors between $\operatorname{mod} K\widetilde{\mathcal{Q}}$ and $\operatorname{mod} K\widetilde{\mathcal{Q}}\#\langle\tilde{\mathbf{a}}\rangle$, and $\widetilde{M}$ and M' for the Morita equivalences

$$\widetilde{M} := EK\mathcal{Q}\#\langle\mathbf{a}\rangle \otimes_{K\mathcal{Q}\#\langle\mathbf{a}\rangle} - : \operatorname{mod} K\mathcal{Q}\#\langle\mathbf{a}\rangle \to \operatorname{mod} K\widetilde{\mathcal{Q}}$$

and

$$M' := E(K\mathcal{Q}\#\langle\mathbf{a}\rangle)\#\langle\tilde{\mathbf{a}}\rangle \otimes_{(K\mathcal{Q}\#\langle\mathbf{a}\rangle)\#\langle\tilde{\mathbf{a}}\rangle} - : \operatorname{mod}(K\mathcal{Q}\#\langle\mathbf{a}\rangle)\#\langle\tilde{\mathbf{a}}\rangle \to \operatorname{mod} K\widetilde{\mathcal{Q}}\#\langle\tilde{\mathbf{a}}\rangle.$$

Then Lemma 3.5.5 applies (with $\Lambda = K\mathcal{Q}\#\langle\mathbf{a}\rangle$ and $G = \langle\tilde{\mathbf{a}}\rangle$), giving a natural isomorphism $M'F' \cong \widetilde{F}\widetilde{M}$. Combining this with Corollary 3.5.3 gives a natural isomorphism

$$\widetilde{F}\widetilde{M} \cong M'F' \cong M'MH. \tag{3.5.4}$$

We note that $M'M : \operatorname{mod} K\mathcal{Q} \to \operatorname{mod} K\widetilde{\mathcal{Q}}\#\langle\tilde{\mathbf{a}}\rangle$ is a Morita equivalence. Thus we can take adjoints above to get $\widetilde{M}F \cong \widetilde{H}M'M$.

In other words, the pairs (F, H) and $(\widetilde{H}, \widetilde{F})$ give rise to the same pair of functors (up to natural isomorphism) between the categories $\operatorname{Rep}(\mathcal{Q}, K)$ and $\operatorname{Rep}(\widetilde{\mathcal{Q}}, K)$.

Corollary 3.5.6. *The ii-indecomposables for $(\mathcal{Q}, \mathbf{a})$ are given up to isomorphism by the images under H of the indecomposables for $\widetilde{\mathcal{Q}}$, and $H(Y) \cong H(Z)$ if and only if $Y \cong {}^{\tilde{\mathbf{a}}^r}Z$ for some r.*

Proof.
If Y is an indecomposable for $\widetilde{\mathcal{Q}}$, then $H(Y)$ is an ii-indecomposable for $(\mathcal{Q}, \mathbf{a})$. Conversely, suppose that X is an indecomposable for $\mathcal{Q}$ and let Y be an indecomposable summand of $F(X)$. Then X is a summand of $H(Y)$ and so all ii-indecomposables for $(\mathcal{Q}, \mathbf{a})$ are obtained up to isomorphism. The last statement follows from Proposition 3.5.4.
■

3.6 Proof of Main Theorem

We are now in a position to complete the proof of Part 1 of Theorem 1.6.1 — namely that given any root $\alpha \in \Delta(\Gamma)_+$, there exists an ii-indecomposable X for $(\mathcal{Q}, \mathbf{a})$ such that $f(\underline{\dim}\, X) = \alpha$.

We shall simplify the notation from the last section and write

$$F : \mathrm{Rep}(\mathcal{Q}, K) \to \mathrm{Rep}(\widetilde{\mathcal{Q}}, K) \quad \text{and} \quad H : \mathrm{Rep}(\widetilde{\mathcal{Q}}, K) \to \mathrm{Rep}(\mathcal{Q}, K) \tag{3.6.1}$$

for the induction and restriction functors.

We have shown that every ii-indecomposable X for $(\mathcal{Q}, \mathbf{a})$ is isomorphic to $H(Y)$, where Y is some indecomposable for $\widetilde{\mathcal{Q}}$. Therefore, to complete the proof, it is enough to show that the map $h : \mathbb{Z}\widetilde{\mathcal{I}} \to \mathbb{Z}\mathcal{I} \to \mathbb{Z}\mathbf{I}$ induced by H and f maps $\Delta(\widetilde{\mathcal{Q}})_+$ onto $\Delta(\Gamma)_+$.

To calculate h, we can restrict ourselves to the semisimple algebras underlying $K\mathcal{Q}$ and $K\widetilde{\mathcal{Q}}$. We recall the isomorphism (3.3.4)

$$\Big(\prod_{\mathbf{i},r} K\varepsilon_{(\mathbf{i},r)}\Big)\#\langle\mathbf{a}\rangle \xrightarrow{\sim} \prod_{\mathbf{i},\mu} \mathbb{M}\,(d_{\mathbf{i}}, K)_{\mu}$$

and that the idempotents $\tilde{\varepsilon}_{(\mathbf{i},\mu)} \in K\widetilde{\mathcal{Q}}$ correspond to the elementary matrices $E(\mathbf{i},\mu)_{00}$. Therefore the map h is given by

$$h(\beta)_{\mathbf{i}} := \sum_{\mu} \beta_{(\mathbf{i},\mu)}. \tag{3.6.2}$$

We first prove a result similar to Lemma 2.1.2, writing $\tilde{e}_{(\mathbf{i},\mu)} \in \mathbb{Z}\widetilde{\mathcal{I}}$ for the simple roots of $\widetilde{\mathcal{Q}}$.

Lemma 3.6.1. *Let* $\tilde{s}_{\mathbf{i}} := \prod_{\mu} \tilde{r}_{(\mathbf{i},\mu)} \in W(\widetilde{\mathcal{Q}})$ *and take* $\beta \in \mathbb{Z}\widetilde{\mathcal{I}}$. *Then*

1. $(h(\beta), e_{\mathbf{i}})_{\Gamma} = d_{\mathbf{i}} \sum_{\mu} (\beta, \tilde{e}_{(\mathbf{i},\mu)})_{\widetilde{\mathcal{Q}}}$;

2. $h(\tilde{s}_{\mathbf{i}}(\beta)) = r_{\mathbf{i}}(h(\beta))$;

3. *the map* $r_{\mathbf{i}} \mapsto \tilde{s}_{\mathbf{i}}$ *induces an isomorphism* $W(\Gamma) \xrightarrow{\sim} C_{\tilde{\mathbf{a}}}(W(\widetilde{\mathcal{Q}}))$.

Proof.

We first note that the element $\tilde{s}_{\mathbf{i}}$ is well-defined since $\tilde{\mathbf{a}}$ is again admissible and, as the bilinear form $(-,-)_{\widetilde{\mathcal{Q}}}$ is $\tilde{\mathbf{a}}$-invariant, $\tilde{s}_{\mathbf{i}}$ commutes with the action of $\tilde{\mathbf{a}}$.

Now, by equation (3.4.5) we have

$$\frac{n}{d_{\mathbf{i}}d_{\mathbf{j}}}b_{\mathbf{ij}} = \tilde{b}_{\mathbf{ij}} = \frac{n}{d_{\mathbf{j}}}\sum_{\mu}\tilde{a}_{(\mathbf{i},\mu)(\mathbf{i},\nu)} \quad \text{for any } \nu.$$

Therefore

$$(h(\beta), e_{\mathbf{i}})_{\Gamma} = \sum_{\mathbf{j}} b_{\mathbf{ij}}h(\beta)_{\mathbf{j}} = d_{\mathbf{i}}\sum_{\mathbf{j},\mu,\nu}\tilde{a}_{(\mathbf{i},\mu)(\mathbf{j},\nu)}\beta_{(\mathbf{j},\nu)} = d_{\mathbf{i}}\sum_{\mu}(\beta, \tilde{e}_{(\mathbf{i},\mu)})_{\widetilde{\mathcal{Q}}}.$$

Also,

$$h(\tilde{s}_{\mathbf{i}}(\beta)) = h(\beta) - \sum_{\mu}(\beta, \tilde{e}_{(\mathbf{i},\mu)})_{\widetilde{\mathcal{Q}}}e_{\mathbf{i}} = h(\beta) - \frac{1}{d_{\mathbf{i}}}(h(\beta), e_{\mathbf{i}})_{\Gamma}e_{\mathbf{i}} = r_{\mathbf{i}}(h(\beta)).$$

Finally, induction on length shows that $C_{\tilde{\mathbf{a}}}(W(\widetilde{\mathcal{Q}}))$ is generated by the $\tilde{s}_{\mathbf{i}}$, so suppose that $r_{\mathbf{i}_1}\cdots r_{\mathbf{i}_m} = 1 \in W(\Gamma)$ and consider $\tilde{s}_{\mathbf{i}_1}\cdots\tilde{s}_{\mathbf{i}_m} \in C_{\tilde{\mathbf{a}}}(W(\widetilde{\mathcal{Q}}))$. For any vertex $(\mathbf{j},\nu) \in \widetilde{\mathcal{I}}$ we must have that $\tilde{s}_{\mathbf{i}_1}\cdots\tilde{s}_{\mathbf{i}_m}(\tilde{e}_{(\mathbf{j},\nu)}) > 0$. This gives that $\ell(\tilde{s}_{\mathbf{i}_1}\cdots\tilde{s}_{\mathbf{i}_m}) = 1$ and hence the $r_{\mathbf{i}}$ and $\tilde{s}_{\mathbf{i}}$ satisfy the same relations.

■

Proposition 3.6.2. *The map* $\beta \mapsto h(\beta)$ *sends* $\Delta(\widetilde{\mathcal{Q}})_+$ *onto* $\Delta(\Gamma)_+$. *Moreover, if* $\alpha \in \Delta(\Gamma)_+$ *is real, then there is a unique* $\tilde{\mathbf{a}}$*-orbit of roots mapping to* α*, all of which are real.*

Proof.

Since the dimension vector of any ii-indecomposable must be a positive root of Γ we know that $\beta \mapsto h(\beta)$ sends $\Delta(\widetilde{\mathcal{Q}})_+$ into $\Delta(\Gamma)_+$. (We could also prove this directly.) To show surjectivity, we first construct preimages for roots in the fundamental region F_Γ by adapting the proof of Lemma 5.3 in [25].

Let $\alpha \in F_\Gamma$ and consider the set $\{\beta \in \Delta(\widetilde{\mathcal{Q}})_+ \mid h(\beta) \leq \alpha\}$. Since this set is finite and non-empty we can take an element β of maximal height. Suppose $h(\beta)_{\mathbf{i}} < \alpha_{\mathbf{i}}$. Then for any μ, $h(\beta + \tilde{e}_{(\mathbf{i},\mu)}) = h(\beta) + e_{\mathbf{i}} \leq \alpha$. By the maximality of β, $\beta + \tilde{e}_{(\mathbf{i},\mu)}$ cannot be a root and so $(\beta, \tilde{e}_{(\mathbf{i},\mu)})_{\widetilde{\mathcal{Q}}} \geq 0$ ([25], Corollary 3.6). Thus $(h(\beta), e_{\mathbf{i}})_\Gamma \geq 0$ as well. In particular, $h(\beta)$ and α must have the same support, for otherwise we can find a vertex $\mathbf{i}$ in the support of α and adjacent to the support of $h(\beta)$, which gives $(h(\beta), e_{\mathbf{i}})_\Gamma < 0$. Contradiction.

We may assume that $\mathrm{supp}(\alpha) = \Gamma$. Let $S := \{\mathbf{i} \mid h(\beta)_{\mathbf{i}} = \alpha_{\mathbf{i}}\}$. If S is the empty set, then $\beta + \tilde{e}_{(\mathbf{i},\mu)}$ is not a root for any vertex of $\widetilde{\mathcal{Q}}$ and so the connected component of $\widetilde{\mathcal{Q}}$ in which β lies is Dynkin ([25], Proposition 4.9). Therefore $\widetilde{\mathcal{Q}}$ must be a disjoint union of copies of this Dynkin quiver, all in a single $\tilde{\mathbf{a}}$ orbit. Then $\widetilde{\Gamma}$ must be connected Dynkin, by the remark following Proposition 2.1.3, and hence Γ must also be Dynkin. Contradiction, since α was assumed to be an imaginary root of Γ.

Thus S is non-empty, so take a connected component T of $\Gamma - S$ and write γ for the restriction of $h(\beta)$ to T. For all vertices $\mathbf{j} \in T$,

$$(\gamma, e_{\mathbf{j}})_T \geq (h(\beta), e_{\mathbf{j}})_\Gamma \geq 0.$$

Moreover, there exists a vertex $\mathbf{j} \in T$ adjacent to S and so $(\gamma, e_{\mathbf{j}})_T > 0$. Therefore T is Dynkin ([25], Corollary 4.3).

Conversely, let γ' be the restriction of $\alpha - h(\beta)$ to T, so γ' has support the whole of T. Then for any vertex $\mathbf{j} \in T$,

$$(\gamma', e_{\mathbf{j}})_T = (\alpha - h(\beta), e_{\mathbf{j}})_\Gamma = (\alpha, e_{\mathbf{j}})_\Gamma - (h(\beta), e_{\mathbf{j}})_\Gamma \leq 0.$$

Hence T is not Dynkin ([25], Theorem 4.3). Contradiction. Therefore $S = \Gamma$ and $h(\beta) = \alpha$.

Clearly every simple root $e_{\mathbf{i}}$ of Γ lies in the image of h and so the correspondence $W(\Gamma) \cong C_{\tilde{\mathbf{a}}}(W(\widetilde{\mathcal{Q}}))$ proves that h is surjective.

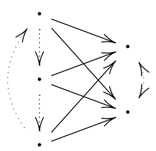
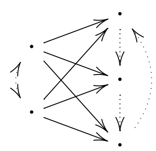
Finally, let $\alpha \in \Delta(\Gamma)^{\mathrm{re}}_+$ and let β be a root for $\widetilde{\mathcal{Q}}$ such that $h(\beta) = \alpha$. Take $w' \in W(\Gamma)$ such that $w'(\alpha)$ is simple, say equal to $e_{\mathbf{i}}$. If w is the corresponding element in $C_{\tilde{\mathbf{a}}}(W(\widetilde{\mathcal{Q}}))$, then $w(\beta)$ must also be simple, equal to some $\tilde{e}_{(\mathbf{i},\mu)}$. Therefore β is real and uniquely determined up to an $\tilde{\mathbf{a}}$-orbit.

■

Theorem 1.6.1 now follows immediately.

3.7 An Example

Using Proposition 3.6.2, we can exhibit a counter-example to the converse of Part 2 of Theorem 1.6.1 — that is, an imaginary root $\alpha \in \Delta(\Gamma)_+$ such that the corresponding ii-indecomposable is unique up to isomorphism. Namely, we consider

$(\mathcal{Q}, \mathbf{a})$  and $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ 

so that Γ is the valued graph $\cdot \overset{(3,2)}{\overline{\quad\quad}} \cdot$. The root $\alpha = (1,1)$ is in the fundamental region for Γ and yet there is a unique $\tilde{\mathbf{a}}$-orbit of roots for $\widetilde{\mathcal{Q}}$ mapping to α, all of which are real. Thus there is a unique ii-indecomposable of dimension vector $f^{-1}(\alpha)$.

Chapter 4

Representations Over Finite Fields

This chapter concerns the number of isomorphism classes of absolutely ii-indecomposable representations of a given dimension vector α over a finite field. We construct a formula for these numbers and show that they are polynomial in the size of the field, provided the field contains a primitive n-th root of unity. Furthermore, the coefficients are rational numbers, are independent of the orientation of the quiver and have denominators bounded by $n\bar{\alpha}$.

We also consider the generating function for the number of isomorphism classes of ii-representations over a finite field. This has an interesting factorisation involving terms coming from each of the subgroups of $\langle \mathbf{a} \rangle$ (equivalently each of the divisors of n).

We fix the following notation. Let K be an algebraically closed field of positive characteristic coprime to n and $(\mathcal{Q}, \mathbf{a})$ a quiver with admissible automorphism of order n. Let $L = \mathbb{F}_q$ be a finite subfield of K containing a primitive n-th root of unity (if and only if $q \equiv 1 \bmod n$). We note that we have an induced action of $\mathbf{a}$ on $L\mathcal{Q}$.

Now let $M \subset K$ be a finite field extension of L. The Galois group Gal of M/L acts naturally on $M\mathcal{Q} \cong M \otimes_L L\mathcal{Q}$ as L-algebra automorphisms. This induces an action of Gal on $M\mathcal{Q}$-modules: ${}^{g}\mathcal{X}$ has the same underlying L-vector space, but with the new action $\lambda \cdot x := g^{-1}(\lambda)x$.

Alternatively, there is an action of Gal on the space $\mathrm{Rep}(\beta, M)$ given by the action of Gal on each co-ordinate. This extends to an action of Gal on the category $\mathrm{Rep}(\mathcal{Q}, M)$. We note that these two actions are naturally isomorphic via the equivalence of categories $\mathrm{Rep}(\mathcal{Q}, M) \cong \mathrm{mod}\, M\mathcal{Q}$.

4.1 Absolutely ii-Indecomposables

Let X be an ii-representation over L. We call X an absolutely ii-indecomposable if the representation $K \otimes_L X$ is ii-indecomposable. Conversely, let $M \subset K$ be a finite field extension of L and X a representation over M. We say that X is defined over L if there exists a representation Y over L such that $M \otimes_L Y \cong X$ over M.

Our aim is to derive a formula relating the numbers of isomorphism classes of absolutely ii-indecomposables and ii-indecomposables (defined over $\mathbb{F}_q$). We use the following notation, where α denotes a positive element of $\mathbb{Z}\mathbf{I}$.

$R\text{-}ii(\alpha, q)$ is the number of isomorphism classes of ii-representations over $\mathbb{F}_q$ of dimension vector $f^{-1}(\alpha)$.

$I\text{-}ii(\alpha, q)$ is the number of isomorphism classes of ii-indecomposables over $\mathbb{F}_q$ of dimension vector $f^{-1}(\alpha)$.

$A\text{-}ii(\alpha, q)$ is the number of isomorphism classes of absolutely ii-indecomposables over $\mathbb{F}_q$ of dimension vector $f^{-1}(\alpha)$.

$M\text{-}ii(\alpha, q, q^t)$ is the number of isomorphism classes of absolutely ii-indecomposables over $\mathbb{F}_{q^t}$ of dimension vector $f^{-1}(\alpha)$, not defined over any subextension of $\mathbb{F}_{q^t}/\mathbb{F}_q$.

We begin with several lemmas about descent theory for finite dimensional modules, following [27]. Let Λ be an L-algebra and M/L a finite field extension.

Then the Galois group Gal acts on $M \otimes_L \Lambda$ as L-algebra automorphisms via $g(m \otimes \lambda) := g(m) \otimes \lambda$. Hence for all $M \otimes_L \Lambda$-modules Y and $g \in \mathrm{Gal}$, we can construct the module ${}^gY$.

Lemma 4.1.1. *Let X be a Λ-module and M/L a field extension of degree r. Then $M \otimes_L X|_L \cong X^r$. In particular, two Λ-modules X and Y are isomorphic if and only if $M \otimes_L X \cong M \otimes_L Y$.*

Proof.
A basis for M/L gives a decomposition of $M \otimes_L X|_L$. The second statement follows from the Krull-Remak-Schmidt Theorem.
■

Lemma 4.1.2. *Let X be a Λ-indecomposable, $M := \mathrm{End}(X)/\mathrm{rad}\,\mathrm{End}(X)$ and $r = [M : L]$. Then $M \otimes_L X$ is a direct sum of r pairwise non-isomorphic indecomposables Y_i with $\mathrm{End}(Y_i)/\mathrm{rad}\,\mathrm{End}(Y_i) = M$.*

Proof.
Since L is perfect, $\mathrm{End}(M \otimes_L X)/\mathrm{rad}\,\mathrm{End}(M \otimes_L X)$ is isomorphic to $M \otimes_L \mathrm{End}(X)/\mathrm{rad}\,\mathrm{End}(X) \cong M \otimes_L M \cong M^r$.
■

Lemma 4.1.3. *Let M/L be a finite field extension with Galois group* Gal*. Let Y be an $M \otimes_L \Lambda$-module such that ${}^gY \cong Y$ for all $g \in \mathrm{Gal}$ and suppose that* Gal *acts transitively on the indecomposable summands of Y (up to isomorphism). Then any indecomposable summand X of $Y|_L$ satisfies $M \otimes_L X \cong Y$.*

Proof.
We have that $M \otimes_L Y \cong \bigoplus_{g \in \mathrm{Gal}} {}^gY \cong Y^r$, using that $M \otimes_L M \cong \bigoplus_{g \in \mathrm{Gal}} {}^gM$ as M-bimodules. Now let X be an indecomposable summand of $Y|_L$. Since ${}^g(M \otimes_L X) \cong M \otimes_L X$ for all $g \in \mathrm{Gal}$, the assumption on Y implies that $M \otimes_L X \cong Y^s$ for some s. If N is a common extension of M and $\mathrm{End}(X)/\mathrm{rad}\,\mathrm{End}(X)$, then on the one hand $N \otimes_L X$ is isomorphic to the direct sum of pairwise non-isomorphic indecomposables, whereas on the other it is isomorphic to $N \otimes_M Y^s$.

Thus $s = 1$.

■

In fact, we now see that if X is a Λ-indecomposable as in Lemma 4.1.2, then Gal must act transitively on the indecomposable summands of $M \otimes_L X$.

Corollary 4.1.4. *There is a 1–1 correspondence between the isomorphism classes of Λ-modules and the isomorphism classes of $M \otimes_L \Lambda$-modules Y such that ${}^gY \cong Y$ for all $g \in \mathrm{Gal}(M/L)$.*

We now return to the case when $\Lambda = L\mathcal{Q}$.

Corollary 4.1.5.
1. *Any splitting field for an indecomposable X over L must contain $M := \mathrm{End}(X)/\mathrm{rad}\,\mathrm{End}(X)$.*
2. *Let M/L be a finite field extension with Galois group* Gal *and Y a representation defined over M. Let $H := \{g \in \mathrm{Gal} \mid {}^gY \cong Y\}$. Then Y is defined over a subextension F of M/L if and only if F contains the fixed field of H.*

Proof.
Let $E \supset L$ be a splitting field for X. Since $M \otimes_L X$ splits as a direct sum of $r = [M : L]$ absolutely indecomposables, $E \otimes_L X$ must also have r summands, independently of E. Moreover, the Galois group $\mathrm{Gal}(E/L)$ must act transitively on these summands. Therefore r divides $[E : L]$ and so $E \supset M$.

We know that Y is defined over F if and only if ${}^gY \cong Y$ for all $g \in \mathrm{Gal}(M/F)$, which occurs if and only if $\mathrm{Gal}(M/F) \leq H$.

■

Recall that $\bar{\alpha}$ denotes the highest common factor of the components $\alpha_{\mathbf{i}}$.

Proposition 4.1.6. *We have the following identities.*

$$A\text{-}ii(\alpha, q^r) = \sum_{t|r} M\text{-}ii(\alpha, q, q^t) \qquad \text{and} \qquad I\text{-}ii(\alpha, q) = \sum_{r|\bar{\alpha}} \frac{1}{r} M\text{-}ii\Big(\frac{\alpha}{r}, q, q^r\Big).$$

Combining these, we obtain

$$A\text{-}ii(\alpha, q) = \sum_{rt|\bar{\alpha}} \frac{1}{rt}\mu(t) I\text{-}ii\Big(\frac{\alpha}{rt}, q^t\Big),$$

where μ is the Möbius function.

Proof.

We can partition the isomorphism classes of absolutely ii-indecomposables over $\mathbb{F}_{q^r}$ according to their minimum fields of definition over $\mathbb{F}_q$, from which we obtain the first formula.

The second formula follows by considering the minimum field over which an ii-indecomposable splits as a direct sum of absolutely ii-indecomposables. Let X be an ii-indecomposable over $L = \mathbb{F}_q$, say $X \cong Y_1 \oplus \cdots \oplus Y_r$ with each Y_i indecomposable over L and ${}^{\mathbf{a}}Y_i \cong Y_{i+1}$. Then all the Y_i have isomorphic endomorphism rings and hence the same minimal splitting field $M = \mathbb{F}_{q^t}$ over L. We can therefore write

$$M \otimes_L Y_i \cong Z_{i,1} \oplus \cdots \oplus Z_{i,t}$$

with each $Z_{i,j}$ absolutely indecomposable over L and ${}^{g}Z_{i,j} \cong Z_{i,j+1}$, for some fixed generator g of $\mathrm{Gal}(M/L)$. Now $\mathbf{a}$ acts on the $Z_{i,j}$ up to isomorphism and without loss of generality we may assume that ${}^{\mathbf{a}}Z_{i,j} \cong Z_{i+1,j}$ for $1 \leq i < r$. Since the actions of $\mathbf{a}$ and g commute, each $\mathbf{a}$-orbit must have the same size, a multiple of r. Therefore, for some $m|t$, the representations

$$W_j := \bigoplus_{i=1}^{r} \bigoplus_{l=1}^{t/m} Z_{i,j+lm} \qquad \text{with } 1 \leq j \leq m$$

are absolutely ii-indecomposable and $X \cong W_1 \oplus \cdots \oplus W_m$. It follows that each W_j has minimum field of definition $\mathbb{F}_{q^m}$ over L and if $\underline{\dim}\, Y_i = \beta_i$, then $\underline{\dim}\, Z_{i,j} = \frac{1}{t}\beta_i$. Therefore $\underline{\dim}\, W_j = \frac{1}{m}\sum_i \beta_i = \frac{1}{m}\underline{\dim}\, X$, from which we deduce that

$$I\text{-}ii(\alpha, q) = \sum_{r|\bar{\alpha}} \frac{1}{r} M\text{-}ii\Big(\frac{\alpha}{r}, q, q^r\Big).$$

Finally, we can invert the first formula to obtain

$$M\text{-}ii(\alpha, q, q^r) = \sum_{t|r} \mu(t) A\text{-}ii(\alpha, q^{r/t}).$$

Substituting this into the second formula gives

$$I\text{-}ii(\alpha,q)=\sum_{r|\bar\alpha}\sum_{t|r}\frac{1}{r}\mu(t)A\text{-}ii\Big(\frac{\alpha}{r},q^{r/t}\Big)=\sum_{rt|\bar\alpha}\frac{1}{rt}\mu(t)A\text{-}ii\Big(\frac{\alpha}{rt},q^{r}\Big).$$

Therefore

$$\begin{aligned}\sum_{rt|\bar\alpha}\frac{1}{rt}\mu(t)I\text{-}ii\Big(\frac{\alpha}{rt},q^{t}\Big)&=\sum_{rt|\bar\alpha}\sum_{su|\bar\alpha/rt}\frac{1}{rstu}\mu(t)\mu(u)A\text{-}ii\Big(\frac{\alpha}{rstu},q^{st}\Big)\\&=\sum_{d|\bar\alpha}\sum_{stu|d}\frac{1}{d}\mu(t)\mu(u)A\text{-}ii\Big(\frac{\alpha}{d},q^{st}\Big)\\&=\sum_{d|\bar\alpha}\sum_{t|d}\frac{1}{d}\mu(t)A\text{-}ii\Big(\frac{\alpha}{d},q^{d}\Big)=A\text{-}ii(\alpha,q),\end{aligned}$$

where we have set $d=rstu$ and successively summed over u then t.

■

4.2 Using Burnside's Lemma

We have now reduced our problem to describing the number of isomorphism classes of ii-indecomposables over the finite field $\mathbb{F}_q$. Also, by the Krull-Remak Schmidt property, Lemma 2.3.1, we have the identity of power series

$$\sum_{\alpha\geq 0}R\text{-}ii(\alpha,q)X^{\alpha}=\prod_{\alpha>0}(1-X^{\alpha})^{-I\text{-}ii(\alpha,q)}. \qquad (4.2.1)$$

We shall now proceed to factorise the left hand side of this equation.

Again, let $0\leq\alpha\in\mathbb{Z}\mathbf{I}$ and write $\beta=f^{-1}(\alpha)$ for the corresponding dimension vector in $(\mathbb{Z}\mathcal{I})^{\langle\mathbf{a}\rangle}$. By applying Burnside's Lemma to the finite group $\mathrm{GL}(\beta,q)$ acting on the finite set $\mathrm{Rep\text{-}ii}(\alpha,q)\subset\mathrm{Rep}(\beta,q)$, we can obtain an expression for the number of isomorphism classes of ii-representations of dimension vector β. Namely

$$R\text{-}ii(\alpha,q)=|\,\mathrm{Rep\text{-}ii}(\alpha,q)/\,\mathrm{GL}(\beta,q)|=\frac{1}{|\,\mathrm{GL}(\beta,q)|}\sum_{g}|\mathrm{Fix}(g)|, \qquad (4.2.2)$$

where $\mathrm{Fix}(g)=\{X\in\mathrm{Rep\text{-}ii}(\alpha,q)\mid g\cdot X=X\}$.

Introducing the sets

$$\mathrm{Fix}'(g) = \{X \in \mathrm{Rep}(\beta, q) \mid g \cdot {}^{\mathbf{a}^{-1}}X = X\} \tag{4.2.3}$$

we have the equality $\sum_g |\mathrm{Fix}(g)| = \sum_g |\mathrm{Fix}'(g)|$, since both sides equal the sum over all representations $X \in \text{Rep-ii}(\alpha, q)$ of the sizes of the stabilisers. Hence we can rewrite (4.2.2) as

$$|\,\text{Rep-ii}(\alpha, q)/\,\mathrm{GL}(\beta, q)| = \frac{1}{|\,\mathrm{GL}(\beta, q)|} \sum_g |\mathrm{Fix}'(g)|. \tag{4.2.4}$$

As in Chapter 3, we can view the path algebra $L\mathcal{Q}$ as a tensor algebra $T(\Lambda_0, \Lambda_1)$. Recall that Λ_0 is a basic semisimple L-algebra and that the $\Lambda_0\#\langle\mathbf{a}\rangle$-bimodule $\Lambda_1\#\langle\mathbf{a}\rangle$ splits as a direct sum of simple modules. Moreover, each of these simples is generated by an arrow of the form $\rho : (\mathbf{i}, 0) \to (\mathbf{j}, l)$ with $\mathbf{a}^t(\rho) = \zeta^{tu}\rho$, where $t = \mathrm{lcm}(d_\mathbf{i}, d_\mathbf{j})$. For simplicity, we shall consider each simple module separately.

Let $\rho : (\mathbf{i}, 0) \to (\mathbf{j}, l)$ be an arrow as above. For any module X, we can consider the corresponding matrices $x_{\mathbf{a}^r(\rho)}$ for $0 \le r < t$. Then the module ${}^{\mathbf{a}^{-1}}X$ will correspond to the matrices $y_{\mathbf{a}^r(\rho)}$, where by formula (2.3.5),

$$y_{\mathbf{a}^r(\rho)} = x_{\mathbf{a}^{r+1}(\rho)} \quad 0 \le r < t-1 \qquad \text{and} \qquad y_{\mathbf{a}^{t-1}(\rho)} = \zeta^{tu} x_\rho. \tag{4.2.5}$$

Thus $X \in \mathrm{Fix}'(g)$ if and only if

$$x_{\mathbf{a}^r(\rho)} = g_{(\mathbf{j},l+r)}\, y_{\mathbf{a}^r(\rho)}\, g^{-1}_{(\mathbf{i},r)} = g_{(\mathbf{j},r+l)}\, x_{\mathbf{a}^{r+1}(\rho)}\, g^{-1}_{(\mathbf{i},r)} \quad \text{for } 0 \le r < t-1 \tag{4.2.6}$$

and

$$x_{\mathbf{a}^{t-1}(\rho)} = \zeta^{tu}\, g_{(\mathbf{j},l+t-1)}\, x_\rho\, g^{-1}_{(\mathbf{i},t-1)}. \tag{4.2.7}$$

That is,

$$\begin{aligned} x_\rho &= \zeta^{tu} \left(g_{(\mathbf{j},l)} g_{(\mathbf{j},l+1)} \cdots g_{(\mathbf{j},l+t-1)}\right) x_\rho \left(g_{(\mathbf{i},0)} g_{(\mathbf{i},1)} \cdots g_{(\mathbf{i},t-1)}\right)^{-1} \\ &= \zeta^{tu} \left(g_{(\mathbf{j},l)} \cdots g_{(\mathbf{j},s+d_\mathbf{j}-1)}\right)^{t/d_\mathbf{j}} x_\rho \left(g_{(\mathbf{i},0)} \cdots g_{(\mathbf{i},d_\mathbf{i}-1)}\right)^{-t/d_\mathbf{i}}. \end{aligned} \tag{4.2.8}$$

Let us write

$$\bar{g}_\mathbf{i} := g_{(\mathbf{i},0)} g_{(\mathbf{i},1)} \cdots g_{(\mathbf{i},d_\mathbf{i}-1)}. \tag{4.2.9}$$

Then, for $h := g_{(\mathbf{j},0)} \cdots g_{(\mathbf{j},l-1)}$, we have that

$$x_\rho \, \bar{g}_{\mathbf{i}}^{t/d_{\mathbf{i}}} = \zeta^{tu} \, h^{-1} \bar{g}_{\mathbf{j}}^{t/d_{\mathbf{j}}} h \, x_\rho. \tag{4.2.10}$$

This is precisely the condition that x_ρ is an $\mathbb{F}_q[T]$-homomorphism

$$V(\bar{g}_{\mathbf{i}}^{t/d_{\mathbf{i}}}) \to V(\zeta^{tu} h^{-1} \bar{g}_{\mathbf{j}}^{t/d_{\mathbf{j}}} h), \tag{4.2.11}$$

where, for a square matrix $N \in \mathbb{M}_n(q)$, $V(N)$ is the $\mathbb{F}_q[T]$-module with underlying vector space ${\mathbb{F}_q}^n$ on which T acts as multiplication by N.

The set of all such homomorphisms $\mathrm{Hom}_{\mathbb{F}_q[T]}(V(N), V(N'))$ is an $\mathbb{F}_q$-vector space with the same dimension as $\mathrm{Hom}_{K[T]}(V(N), V(N'))$, where $K = \overline{\mathbb{F}}_q$. Furthermore, this dimension depends only on the conjugacy classes of N and N'.

Summing up, for each simple summand of the $\Lambda_0\#\langle\mathbf{a}\rangle$-bimodule $\Lambda_1\#\langle\mathbf{a}\rangle$ we can take a generator $\rho : (\mathbf{i}, 0) \to (\mathbf{j}, l)$ with $\mathbf{a}^t(\rho) = \zeta^{tu}\rho$. We then get a contribution to $|\mathrm{Fix}'(g)|$ of a factor

$$q^{\dim \mathrm{Hom}_{K[T]} \left(V(\bar{g}_{\mathbf{i}}^{t/d_{\mathbf{i}}}), V(\zeta^{tu} \bar{g}_{\mathbf{j}}^{t/d_{\mathbf{j}}}) \right)}. \tag{4.2.12}$$

In particular, $|\mathrm{Fix}'(g)|$ for $(g_{(\mathbf{i},r)}) \in \mathrm{GL}(\beta, q)$ depends only on the conjugacy class of the corresponding element $(\bar{g}_{\mathbf{i}}) \in \mathrm{GL}(\alpha, q)$. Also, given $(\bar{g}_{\mathbf{i}}) \in \mathrm{GL}(\alpha, q)$, there are $\prod_{\mathbf{i}} |\mathrm{GL}_{\alpha_{\mathbf{i}}}(q)|^{d_{\mathbf{i}}-1}$ choices for $(g_{(\mathbf{i},r)}) \in \mathrm{GL}(\beta, q)$ such that (4.2.9) holds. Therefore we can write

$$R\text{-}ii(\alpha, q) = \frac{1}{|\mathrm{GL}(\alpha, q)|} \sum_{\bar{g} \in \mathrm{GL}(\alpha,q)} |\mathrm{Fix}'(\bar{g})| = \sum_{\bar{g} \in \mathcal{C}(\alpha,q)} \frac{|\mathrm{Fix}'(\bar{g})|}{|Z(\bar{g})|}, \tag{4.2.13}$$

where $\mathcal{C}(\alpha, q)$ is a set of representatives for the conjugacy classes in $\mathrm{GL}(\alpha, q)$ and $Z(\bar{g})$ is the centraliser of $\bar{g}$ in $\mathrm{GL}(\alpha, q)$ (c.f. the formula on page 1015 of [19]).

4.3 Conjugacy Classes of $\mathrm{GL}(\alpha, q)$

In this section we describe a set of representatives for the conjugacy classes of $\mathrm{GL}(\alpha, q)$ and calculate their contributions to $R\text{-}ii(\alpha, q)$.

A partition is a sequence $\lambda = (1^{\lambda_1}2^{\lambda_2}\cdots)$ with almost all $\lambda_i = 0$. We denote the set of all partitions, including the unique partition of 0, by $\mathcal{P}$. Given any partitions $\lambda = (1^{\lambda_1}\cdots)$ and $\mu = (1^{\mu_1}\cdots)$, we write $|\lambda| = \sum_i i\lambda_i$ and set

$$\langle\lambda,\mu\rangle := \sum_{i,j}\min(i,j)\lambda_i\mu_j. \tag{4.3.1}$$

Let Φ be the set of all monic irreducible polynomials in $\mathbb{F}_q[T]$ except T itself and write Φ_r for the subset consisting of those polynomials of degree r. For any $f = T^r - \sum_{i=1}^r f_iT^{i-1} \in \Phi$ we let $J(f)$ be its companion matrix — that is, the $r\times r$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ f_1 & f_2 & f_3 & \cdots & f_r \end{pmatrix}.$$

N.B. If $f = T - a$, then we also write $J(f) = J(a)$.

Now, for $m \geq 1$, we denote by $J_m(f)$ the $mr\times mr$ matrix with $J(f)$ on the diagonal and the identity matrix I_r on the upper diagonal. Finally, for any partition $\lambda = (1^{\lambda_1}2^{\lambda_2}\cdots)$, we set

$$J(f,\lambda) := J_1(f)^{\oplus\lambda_1} \oplus J_2(f)^{\oplus\lambda_2} \oplus \cdots\,. \tag{4.3.2}$$

Proposition 4.3.1. *The conjugacy classes in* $\mathrm{GL}(n,q)$ *can be represented by the matrices*

$$J(f_1,\lambda_1)\oplus J(f_2,\lambda_2)\oplus\cdots\,,$$

where the f_i *are distinct polynomials in* Φ*, the* λ_i *are partitions in* $\mathcal{P}$ *and* $\sum_i \deg(f_i)|\lambda_i| = n$*.*

Moreover, for $f\in\Phi_r$*, the size of the centraliser* $Z(J(f,\lambda))$ *is given by the formula*

$$|Z(J(f,\lambda))| = c(\lambda,q^r) := q^{r\langle\lambda,\lambda\rangle}\prod_i\prod_{m=1}^{\lambda_i}(1-q^{-rm}).$$

Proof.
This can be found in [31]. We note that $c(\lambda, q) \in \mathbb{Z}[q]$.
■

Let $\mathbf{i}$ and $\mathbf{j}$ be vertex orbits of sizes d and e respectively and set $t = \operatorname{lcm}(d, e)$. We fix $\rho : (\mathbf{i}, 0) \to (\mathbf{j}, l)$ with $\mathbf{a}^t(\rho) = \zeta^{tu}\rho$ and consider for $f, g \in \Phi$ and $\lambda, \mu \in \mathcal{P}$

$$\dim \operatorname{Hom}_{K[T]} \Big(V\big(J(f,\lambda)^{t/d}\big), V\big(\zeta^{tu} J(g,\mu)^{t/e}\big)\Big). \tag{4.3.3}$$

Suppose that f and g have degrees r and s respectively and that they split over K as

$$f = \prod_{i=1}^{r}(T - \xi_i), \qquad g = \prod_{j=1}^{s}(T - \eta_j).$$

We know that over K, $J(f,\lambda)$ is conjugate to $J(\xi_1,\lambda) \oplus \cdots \oplus J(\xi_r,\lambda)$. Also, since q and n are coprime, $J(f,\lambda)^{t/d}$ is conjugate to $J(\xi_1^{t/d},\lambda) \oplus \cdots \oplus J(\xi_r^{t/d},\lambda)$. Similarly, $\zeta^{tu}J(g,\mu)^{t/e}$ is conjugate to $J(\zeta^{tu}\eta_1^{t/e},\lambda) \oplus \cdots \oplus J(\zeta^{tu}\eta_s^{t/e},\lambda)$.

Therefore

$$\begin{aligned}\operatorname{Hom}_{K[T]} &\Big(V\big(J(f,\lambda)^{t/d}\big), V\big(\zeta^{tu} J(g,\mu)^{t/e}\big)\Big) \\ &\cong \bigoplus_{i,j} \Big(\bigoplus_{v=1}^{r} \bigoplus_{w=1}^{s} \operatorname{Hom}_{K[T]} \big(V(J_i(\xi_v^{t/d})), V(J_j(\zeta^{tu}\eta_w^{t/e}))\big)\Big)^{\lambda_i\mu_j}. \end{aligned} \tag{4.3.4}$$

In particular, this is zero unless $\xi_v^{t/d} = \zeta^{tu}\eta_w^{t/e}$ for some v and w.

Lemma 4.3.2. *We have $\xi_v^{t/d} = \zeta^{tu}\eta_w^{t/e}$ only if $\xi_v^{n/d} = \eta_w^{n/e}$. This latter condition is equivalent to there being $h \in \Phi$ such that $f(T)|h(T^{n/d})$ and $g(T)|h(T^{n/e})$.*

Proof.
We may assume that $\xi_i^q = \xi_{i+1}$ (with the indices being taken modulo r). Let $\theta_i = \xi_i^{n/d}$ for $1 \le i \le m$ be the distinct n/d-th powers of the ξ_i (so $m|r$) and set

$$h = (T - \theta_1)\cdots(T - \theta_m).$$

Then $h \in \Phi$ since the root θ_1 has order m under the Frobenius map $x \mapsto x^q$ and, since ξ_1 is a root of $h(T^{n/d})$, f divides $h(T^{n/d})$.
■

We introduce the following notation. For two integers s and t let

$$(s,t) := \mathrm{hcf}(s,t) \qquad \text{and} \qquad [s,t] := \mathrm{lcm}(s,t). \tag{4.3.5}$$

Consider first the case $h(T) = T - \theta$. Let $\xi \in K$ be an n-th root of θ. Then ξ^q/ξ is a primitive m-th root of unity for some $m|n$, independent of the choice of ξ, and $m|n$ is minimal such that $\xi^m \in \mathbb{F}_q$. For $d|n$, set

$$h_{d,r}(T) := T^{[m,d]/d} - \zeta^{r[m,d]}\xi^{[m,d]} = \prod_{v=1}^{[m,d]/d} \left(T - \zeta^{rd+vnd/[m,d]}\xi^d\right). \tag{4.3.6}$$

This lies in Φ since the root $\zeta^{rd}\xi^d$ has order $[m,d]/d$ under the Frobenius map and so we have the factorisation

$$h(T^{n/d}) = \prod_{r=1}^{n/[m,d]} h_{d,r}(T) \tag{4.3.7}$$

as a product of irreducibles over $\mathbb{F}_q$.

Remark 4.3.3. *We observe that the number n/m is also determined by the number of irreducible factors of $h(T^n)$.*

Let $f = h_{d,r}$ and consider what happens when each root of f is raised to the power t/d. In general, if we raise the a-th roots of unity to the power b, then we obtain the $[a,b]/b = a/(a,b)$-th roots of unity, each with multiplicity (a,b). In our example, $a = [m,d]/d$ and $b = t/d$, so $[a,b] = [m,t]/d$ and $(a,b) = (m,t)/(m,d)$. Therefore, when we raise each root of f to the power t/d, we get

$$\prod_{v=1}^{[m,t]/t} \left(T - \zeta^{rt+vnt/[m,t]}\xi^t\right)^{\frac{(m,t)}{(m,d)}} = \left(T^{[m,t]/t} - \zeta^{r[m,t]}\xi^{[m,t]}\right)^{\frac{(m,t)}{(m,d)}} = h_{t,r}(T)^{\frac{(m,t)}{(m,d)}}. \tag{4.3.8}$$

That is, we obtain the roots of $h_{t,r}$ each with multiplicity $(m,t)/(m,d)$. Thus, writing $\sim$ for conjugacy of matrices over K, we have

$$J(h_{d,r},\lambda)^{t/d} \sim \bigoplus_{\substack{\xi \text{ root} \\ \text{of } h_{d,r}}} J(\xi^{t/d},\lambda) = \bigoplus_{\substack{\xi \text{ root} \\ \text{of } h_{t,r}}} J(\xi,\lambda)^{\oplus\frac{(m,t)}{(m,d)}} \sim J(h_{t,r},\lambda)^{\oplus\frac{(m,t)}{(m,d)}}. \tag{4.3.9}$$

Similarly if $g = h_{e,s}$, then when we raise each root of g to the power t/e we get $h_{t,s}(T)^{\frac{(m,t)}{(m,e)}}$. Also, if we multiply each root of $h_{t,s}$ by ζ^{tu}, then this gives

$$\prod_{v=1}^{[m,t]/t} \left(T - \zeta^{(s+u)t+vnt/[m,t]}\eta^t\right) = T^{[m,t]/t} - \zeta^{(s+u)[m,t]}\eta^{[m,t]} = h_{t,s+u}(T). \quad (4.3.10)$$

Therefore

$$\zeta^{tu} J(h_{e,s}, \mu)^{t/e} \sim \bigoplus_{\substack{\eta \text{ root} \\ \text{of } h_{e,s}}} J(\zeta^{tu}\eta^{t/e}, \mu) = \bigoplus_{\substack{\eta \text{ root} \\ \text{of } h_{t,s+u}}} J(\eta, \mu)^{\oplus\frac{(m,t)}{(m,e)}} \sim J(h_{t,s+u}, \mu)^{\oplus\frac{(m,t)}{(m,e)}}. \quad (4.3.11)$$

In particular, for $f = h_{d,r}$ and $g = h_{e,s}$ we have

$$\begin{aligned} &\dim \operatorname{Hom}_{K[T]} \Big(V\big(J(f,\lambda)^{t/d}\big), V\big(\zeta^{tu} J(g,\mu)^{t/e}\big)\Big) \\ &\qquad = \frac{(m,t)}{(m,d)} \frac{(m,t)}{(m,e)} \dim \operatorname{Hom}_{K[T]} \big(V(J(h_{t,r},\lambda)), V(J(h_{t,s+u},\mu))\big). \end{aligned} \quad (4.3.12)$$

For any $f, g \in \Phi$ and $\lambda, \mu \in \mathcal{P}$ we have the formula ([19], Lemma 3.3)

$$\dim \operatorname{Hom}_{K[T]} \big(V(J(f,\lambda)), V(J(g,\mu))\big) = \begin{cases} \deg(f)\langle\lambda,\mu\rangle & \text{if } f = g; \\ 0 & \text{otherwise.} \end{cases} \quad (4.3.13)$$

Thus, since $\deg(h_{t,r}) = m/(m,t)$, (4.3.12) equals

$$\frac{(m,t)}{(m,d)} \frac{(m,t)}{(m,e)} \frac{m}{(m,t)} \langle\lambda,\mu\rangle \quad \text{if } r \equiv s+u \bmod n/[m,t]; \qquad 0 \text{ otherwise.} \quad (4.3.14)$$

Summing up, we have

Lemma 4.3.4. *Let* $\mathbf{i}, \mathbf{j}$ *be vertex orbits of respective sizes* d, e*. Let* $t = [d,e]$ *and* $\rho : (\mathbf{i}, 0) \to (\mathbf{j}, l)$ *an arrow such that* $\mathbf{a}^t(\rho) = \zeta^{tu}\rho$*. Fix* $h = T - \theta \in \Phi$ *such that* $h(T^n)$ *has* n/m *irreducible factors over* $\mathbb{F}_q$*. Then we get a contribution towards* $|\mathrm{Fix}'(J(h_{d,r},\lambda), J(h_{e,s},\mu))|$ *of*

$$q^{\frac{m(m,t)}{(m,d)(m,e)}\langle\lambda,\mu\rangle} \quad \textit{if } r \equiv s+u \bmod n/[m,t]; \qquad \textit{0 otherwise.}$$

Now suppose that $h \in \Phi$. By working over the splitting field for h we can write

$$h(T) = \prod_{i=1}^{\deg(h)} (T - \theta_i)$$

with $\theta_i^q = \theta_{i+1}$. Let $\xi_i \in K$ be an n-th root of θ_i. Then $\xi_i^{q^{\deg(h)}}/\xi_i$ is again a primitive m-th root of unity for some $m|n$, independent of both i and the choice of ξ_i. We fix an n-th root ξ_1 of θ_1 and put $\xi_{i+1} = \xi_i^q$ for $1 \leq i < \deg(h)$.

If $d|n$, then

$$h_{d,r}(T) := \prod_i \left(T^{[m,d]/d} - \zeta^{r[m,d]}\xi_i^{[m,d]}\right) \tag{4.3.15}$$

is irreducible over $\mathbb{F}_q$ since the root $\zeta^{rd}\xi_1^d$ has order $\deg(h)[m,d]/d$ under the Frobenius map. Therefore we have the factorisation

$$h(T^{n/d}) = \prod_{r=1}^{n/[m,d]} h_{d,r}(T) \tag{4.3.16}$$

as a product of irreducibles over $\mathbb{F}_q$.

Now consider raising the roots of $h_{d,r}$ to the power t/d. By looking at each factor $T^{[m,d]/d} - \zeta^{r[m,d]}\xi_i^{[m,d]}$ separately, the same argument as before shows that we obtain $h_{t,r}(T)^{(m,t)/(m,d)}$.

Lemma 4.3.5. *Let $\mathbf{i}, \mathbf{j}$ be vertex orbits of respective sizes d, e. Let $t = [d, e]$ and $\rho : (\mathbf{i}, 0) \to (\mathbf{j}, l)$ an arrow such that $\mathbf{a}^t(\rho) = \zeta^{tu}\rho$. Fix $h \in \Phi$ such that $h(T^n)$ has n/m irreducible factors over $\mathbb{F}_q$. Then we get a contribution towards $|\mathrm{Fix}'(J(h_{d,r}, \lambda), J(h_{e,s}, \mu))|$ of*

$$q^{\deg(h)\frac{m(m,t)}{(m,d)(m,e)}\langle\lambda,\mu\rangle} \quad \text{if } r \equiv s + u \bmod n/[m,t]; \qquad \text{0 otherwise.}$$

4.4 A Factorisation

For $m|n$ let $\mathbf{I}(m)$ be the set of pairs $(\mathbf{i}, r)$ where $\mathbf{i} \in \mathbf{I}$ and $0 \leq r < n/[m, d_\mathbf{i}]$. We define a symmetric matrix $B(m)$ indexed by $\mathbf{I}(m)$ as follows.

We write

$$b'(m)_{(\mathbf{i},r)(\mathbf{j},s)} := -(m,t) \sum_{\substack{\rho:(\mathbf{i},0)\to(\mathbf{j},l) \\ \mathbf{a}^t(\rho)=\zeta^{tu}\rho}} 1_{\{r\equiv s+u \bmod n/[m,t]\}}, \tag{4.4.1}$$

where we are summing over generators for each of the simple submodules of $\Lambda_1\#\langle\mathbf{a}\rangle$, and then set

$$b(m)_{(\mathbf{i},r)(\mathbf{j},s)} := b'(m)_{(\mathbf{i},r)(\mathbf{j},s)} + b'(m)_{(\mathbf{j},s)(\mathbf{i},r)}. \tag{4.4.2}$$

We also choose an ordering of the set $\mathbf{I}(m)$ and let $\Phi(m)_r$ to be those $h \in \Phi_r$ such that $h(T^n)$ splits as a product of n/m irreducible polynomials.

Proposition 4.4.1. *For $h \in \Phi(m)$ and partitions $\lambda_{(\mathbf{i},r)}$ with $(\mathbf{i},r) \in \mathbf{I}(m)$ consider*

$$\bar{g}_{\mathbf{i}} := \bigoplus_{r=1}^{n/[m,d_{\mathbf{i}}]} J(h_{d_{\mathbf{i}},r}, \lambda_{(\mathbf{i},r)}).$$

Write $\bar{g} = (\bar{g}_{\mathbf{i}}) \in \mathrm{GL}(\alpha,q)$, where $\alpha_{\mathbf{i}} = \deg(h)\frac{m}{(m,d_{\mathbf{i}})}\sum_r |\lambda_{(\mathbf{i},r)}|$. Then

$$|\mathrm{Fix}'(\bar{g})| = q^{-\deg(h)\sum_{(\mathbf{i},r)<(\mathbf{j},s)} \frac{m}{(m,d_{\mathbf{i}})(m,d_{\mathbf{j}})} b(m)_{(\mathbf{i},r)(\mathbf{j},s)}\langle\lambda_{(\mathbf{i},r)},\lambda_{(\mathbf{i},s)}\rangle}$$

and

$$|Z(\bar{g}_{\mathbf{i}})| = \prod_r c\big(\lambda_{(\mathbf{i},r)}, q^{\deg(h)m/(m,d_{\mathbf{i}})}\big).$$

On the other hand, if $\bar{g}$ and $\bar{g}'$ are given in terms of distinct $h, h' \in \Phi$, then

$$|\mathrm{Fix}'(\bar{g}\oplus\bar{g}')| = |\mathrm{Fix}'(\bar{g})||\mathrm{Fix}'(\bar{g}')|.$$

Proof.
This is immediate from the decomposition of $A_1\#\langle\mathbf{a}\rangle$ into a direct sum of simple $A_0\#\langle\mathbf{a}\rangle$-bimodules and the results of the previous section.
■

For $m|n$ we now define a power series

$$P_m(X,q) := \sum_{\lambda_{(\mathbf{i},r)}\in\mathcal{P}} q^{-\sum_{(\mathbf{i},r)<(\mathbf{j},s)} \frac{m}{(m,d_{\mathbf{i}})(m,d_{\mathbf{j}})} b(m)_{(\mathbf{i},r)(\mathbf{j},s)}\langle\lambda_{(\mathbf{i},r)},\lambda_{(\mathbf{j},s)}\rangle} \times \prod_{(\mathbf{i},r)} \frac{X_{\mathbf{i}}^{(m/(m,d_{\mathbf{i}}))|\lambda_{(\mathbf{i},r)}|}}{c\big(\lambda_{(\mathbf{i},r)}, q^{m/(m,d_{\mathbf{i}})}\big)}. \tag{4.4.3}$$

(C.f. the power series on page 1020 of [19].) We note that this power series is independent of the orientation of the quiver $\mathcal{Q}$.

Theorem 4.4.2. *We have the following factorisation*

$$\sum_{\alpha\geq 0} R\text{-}ii(\alpha,q)X^\alpha = \prod_{\alpha>0}(1-X^\alpha)^{-I\text{-}ii(\alpha,q)} = \prod_{m|n}\prod_{r\geq 1} P_m(X^r,q^r)^{|\Phi(m)_r|}.$$

Proof.

This follows immediately from the identities (4.2.1) and (4.2.13) as well as the previous proposition.

■

We also show that the numbers $|\Phi(m)_r|$ are rational polynomials in q.

Let $\mathbb{F}_{q^r}^{\times m} := \{\theta^m \mid 0 \neq \theta \in \mathbb{F}_{q^r}\}$ and for any $t|r$ set

$$\begin{aligned} C_{r,m} &:= \{\theta \in \mathbb{F}_{q^r}^{\times n/m} \mid \theta \notin \mathbb{F}_{q^r}^{\times n/m'} \text{ for } m'|n \text{ and } m' < m\} \\ \Theta_{r,t,m} &:= \{\theta \in C_{r,m} \mid \theta \in \mathbb{F}_{q^t}\} \\ \Psi_{r,t,m} &:= \{\theta \in C_{r,m} \mid \mathbb{F}_q[\theta] = \mathbb{F}_{q^t}\}. \end{aligned} \tag{4.4.4}$$

Let $\theta \in \mathbb{F}_{q^r}$, let h be its minimal polynomial over $\mathbb{F}_q$ and let ξ be an n-th root of θ in K. Then ξ^{q^r}/ξ is an m-th root of unity if and only if $\xi^m \in \mathbb{F}_{q^r}$, which is if and only if $\theta \in \mathbb{F}_{q^r}^{\times n/m}$. Hence

$$|\Psi_{r,r,m}| = r|\Phi(m)_r|. \tag{4.4.5}$$

Also, $\Theta_{r,t,m} = \dot{\bigcup}_{s|t}\Psi_{r,s,m}$, so that $|\Theta_{r,t,m}| = \sum_{s|t}|\Psi_{r,s,m}|$. Inverting this gives

$$|\Psi_{r,t,m}| = \sum_{s|t}\mu(s)|\Theta_{r,t/s,m}|. \tag{4.4.6}$$

Lastly,

$$|C_{r,m}| = \sum_{s|m}\mu(m/s)|\mathbb{F}_{q^r}^{\times n/s}| = \frac{q^r-1}{n}\sum_{s|m}\mu(m/s)s = \frac{q^r-1}{n}\phi(m), \tag{4.4.7}$$

where ϕ is Euler's Phi function.

Suppose that $\theta \in C_{r/t,m}$ and let $\xi \in K$ be an n-th root of θ. We know that $\xi^{q^{r/t}}/\xi = \eta$ is a primitive m-th root of unity and thus $\xi^{q^r}/\xi = \eta^t$ is a primitive $m/(m,t)$-th root of unity. Hence $\theta \in C_{r,m/(m,t)}$. Therefore

$$\Theta_{r,r/t,m} = \dot{\bigcup_{\substack{s|n/m \\ (ms,t)=s}}} C_{r/t,ms} = \dot{\bigcup_{\substack{s|(n/m,t) \\ (m,t/s)=1}}} C_{r/t,ms} \tag{4.4.8}$$

and thus

$$|\Theta_{r,r/t,m}| = \frac{q^{r/t}-1}{n} \sum_{\substack{s|(n/m,t)\\(m,t/s)=1}} \phi(ms). \tag{4.4.9}$$

We use the following notation. For a prime p, let $t_{[p]} := p^{\mathrm{ord}_p(t)}$ be the p-th part of the prime factorisation of t, and more generally, for an integer m, set $t_{[m]} = \prod_{p|m} t_{[p]}$. Then

$$\sum_{\substack{s|(n/m,t)\\(m,t/s)=1}} \phi(ms) = 1_{\{t_{[m]}|n/m\}} \sum_{r|(n/m,t)/t_{[m]}} \phi(mrt_{[m]}), \tag{4.4.10}$$

since if $(m, t/s) = 1$, then $t_{[m]}$ must divide s, which in turn divides n/m.

We now note that r and $mt_{[m]}$ must be coprime and, since ϕ is multiplicative, $\phi(mrt_{[m]}) = \phi(mt_{[m]})\phi(r)$. Therefore (4.4.10) becomes

$$1_{\{t_{[m]}|n/m\}}\phi(mt_{[m]}) \sum_{r|(n/m,t)/t_{[m]}} \phi(r) = 1_{\{t_{[m]}|n/m\}}\phi(mt_{[m]})\frac{(n/m,t)}{t_{[m]}} =: N(t,m). \tag{4.4.11}$$

We note that $N(t,m)$ is a non-negative integer.

Finally,

$$\begin{aligned} |\Phi(m)_r| = \frac{1}{r}|\Psi_{r,r,m}| &= \frac{1}{r}\sum_{t|r}\mu(t)|\Theta_{r,r/t,m}| \\ &= \frac{1}{rn}\sum_{t|r}\mu(t)(q^{r/t}-1)N(t,m) =: \varphi_{r,m}(q). \end{aligned} \tag{4.4.12}$$

It is now possible to derive an explicit formula for the numbers $A\text{-}ii(\alpha, q)$.

Let us write

$$P_m(X,q) = 1 + \sum_{\alpha>0} \mathrm{p}_m(\alpha,q)X^\alpha \quad \text{and} \quad \log P_m(X,q) = \sum_{\alpha>0} \mathrm{lp}_m(\alpha,q)X^\alpha. \tag{4.4.13}$$

Then

$$\log \prod_{m|n}\prod_{r\geq 1} P_m(X^r,q^r)^{\varphi_{r,m}(q)} = \sum_{m|n}\sum_{\alpha>0}\sum_{r\geq 1}\varphi_{r,m}(q)\mathrm{lp}_m(\alpha,q^r)X^{r\alpha}. \tag{4.4.14}$$

On the other hand, we also have that

$$\log \prod_{\alpha>0} (1 - X^{\alpha})^{-I\text{-}ii(\alpha,q)} = \sum_{\alpha>0} \sum_{r\geq 1} \frac{1}{r} I\text{-}ii(\alpha, q) X^{r\alpha}. \tag{4.4.15}$$

These latter two identities are equal by Theorem 4.4.2 and so we can compare coefficients of X^{α}. That is,

$$\sum_{r|\bar{\alpha}} \frac{1}{r} I\text{-}ii\Big(\frac{\alpha}{r}, q\Big) = \sum_{m|n} \sum_{r|\bar{\alpha}} \varphi_{r,m}(q) \mathrm{lp}_m\Big(\frac{\alpha}{r}, q^r\Big), \tag{4.4.16}$$

from which we get

$$\begin{aligned} I\text{-}ii(\alpha, q) &= \sum_{t|\bar{\alpha}} \frac{1}{t} \mu(t) \sum_{m|n} \sum_{r|\bar{\alpha}/t} \varphi_{r,m}(q) \mathrm{lp}_m\Big(\frac{\alpha}{rt}, q^r\Big) \\ &= \sum_{m|n} \sum_{rt|\bar{\alpha}} \frac{1}{t} \mu(t) \varphi_{r,m}(q) \mathrm{lp}_m\Big(\frac{\alpha}{rt}, q^r\Big) \end{aligned} \tag{4.4.17}$$

and hence

$$\begin{aligned} A\text{-}ii(\alpha, q) &= \sum_{rt|\bar{\alpha}} \frac{1}{rt} \mu(t) I\text{-}ii\Big(\frac{\alpha}{rt}, q^t\Big) \\ &= \sum_{m|n} \sum_{rt|\bar{\alpha}} \sum_{su|\bar{\alpha}/rt} \frac{1}{rtu} \mu(t)\mu(u) \varphi_{s,m}(q^t) \mathrm{lp}_m\Big(\frac{\alpha}{rstu}, q^{st}\Big) \\ &= \sum_{m|n} \sum_{d|\bar{\alpha}} \sum_{stu|d} \frac{s}{d} \mu(t)\mu(u) \varphi_{s,m}(q^t) \mathrm{lp}_m\Big(\frac{\alpha}{d}, q^{st}\Big) \\ &= \sum_{m|n} \sum_{d|\bar{\alpha}} \sum_{t|d} \frac{1}{t} \mu(t) \varphi_{d/t,m}(q^t) \mathrm{lp}_m\Big(\frac{\alpha}{d}, q^d\Big) \end{aligned} \tag{4.4.18}$$

by first setting $d = rstu$ and then summing over $u|d/st$.

Now, by formula (4.4.12),

$$\begin{aligned} \sum_{t|d} \frac{1}{t} \mu(t) \varphi_{d/t,m}(q^t) &= \sum_{t|d} \frac{1}{t} \mu(t) \sum_{r|d/t} \frac{t}{dn} \mu(r)(q^{d/r} - 1) N(r, m) \\ &= \frac{1}{dn} \sum_{rt|d} \mu(r)\mu(t)(q^{d/r} - 1) N(r, m) = \frac{q-1}{dn} \mu(d) N(d, m). \end{aligned} \tag{4.4.19}$$

Therefore,

Proposition 4.4.3.

$$A\text{-}ii(\alpha, q) = \frac{q-1}{n} \sum_{m|n} \sum_{d|\bar{\alpha}} \frac{1}{d} \mu(d) N(d, m) \mathrm{lp}_m\Big(\frac{\alpha}{d}, q^d\Big).$$

This generalises the formula on page 1023 of [19].

We now wish to show that the $A\text{-}ii(\alpha, q)$ are rational polynomials.

Lemma 4.4.4. *We have the identities*

$$\mathrm{lp}_m(\alpha, q) = \sum_{\substack{\sum r_i\beta_i = \alpha \\ \beta_i\, distinct}} (-1)^{\sum r_i} \Big(\sum r_i - 1\Big)! \prod_i \frac{1}{r_i!} \mathrm{p}_m(\beta_i, q)^{r_i}$$

and conversely

$$\mathrm{p}_m(\alpha, q) = \sum_{\substack{\sum r_i\beta_i = \alpha \\ \beta_i\, distinct}} \prod_i \frac{1}{r_i!} \mathrm{lp}_m(\beta_i, q)^{r_i}.$$

Proof.
These follow from the identities

$$\log\Big(1 + \sum_{\alpha>0} \mathrm{p}_m(\alpha, q) X^\alpha\Big) = \sum_{r\geq 1} \frac{(-1)^r}{r} \Big(\sum_{\alpha>0} \mathrm{p}_m(\alpha, q) X^\alpha\Big)^r$$

and

$$\exp\Big(\sum_{\alpha>0} \mathrm{lp}_m(\alpha, q) X^\alpha\Big) = 1 + \sum_{r>0} \frac{1}{r!} \Big(\sum_{\alpha>0} \mathrm{lp}_m(\alpha, q) X^\alpha\Big)^r.$$

■

By definition, the coefficients $\mathrm{p}_m(\alpha, q)$ are all of the form f/g with $f, g \in \mathbb{Z}[q]$ and g monic. Also, for $r = r_1 + \cdots + r_t$, we note that $\frac{(r-1)!}{r_1!\cdots r_t!} \in \frac{1}{r_i}\mathbb{Z}$ for each $1 \leq i \leq t$ and hence the denominator of $\frac{(r-1)!}{r_1!\cdots r_t!}$ is a divisor of $d = \mathrm{hcf}(r_1, \ldots, r_t)$. Since $\alpha = \sum_i r_i\beta_i$, we see that d divides $\bar{\alpha}$. Thus each $\mathrm{lp}_m(\alpha, q)$ is of the form $\frac{1}{\bar{\alpha}} f/g$ with $f, g \in \mathbb{Z}[q]$ and g monic.

From this it is clear that $A\text{-}ii(\alpha, q)$ is a polynomial in $\frac{1}{n\bar{\alpha}}\mathbb{Z}[q]$. For, it can certainly be expressed in the form $\frac{1}{n\bar{\alpha}} f/g$ for some polynomials $f, g \in \mathbb{Z}[q]$ with g monic and since $A\text{-}ii(\alpha, q)$ takes integer values for all prime powers $q \equiv 1 \bmod n$, it must in fact be a polynomial.

We have thus proved

Theorem 4.4.5. *The numbers $I\text{-}ii(\alpha, q)$ and $A\text{-}ii(\alpha, q)$ for q a prime power congruent to 1 modulo n are polynomials in $\frac{1}{n\bar{\alpha}}\mathbb{Z}[q]$ with coefficients that are independent of the orientation of the quiver.*

As an immediate corollary we deduce that the numbers $R\text{-}ii(\alpha, q)$ are also polynomials in q with rational coefficients which are independent of the orientation of the quiver.

4.5 Invariance under the Weyl Group

In this section we use the reflection functors $S_i^{\pm}$ introduced in Chapter 2 to prove that the polynomials $A\text{-}ii(\alpha, q)$ are invariant under the action of the Weyl group $W(\Gamma)$, thus completing the proof of Theorem 1.6.2.

Proposition 4.5.1. *Let $e_{\mathbf{i}} \neq \alpha \in \mathbb{Z}\mathbf{I}$ be positive. Then the polynomials $A\text{-}ii(\alpha, q)$ and $A\text{-}ii(r_{\mathbf{i}}(\alpha), q)$ are equal.*

Proof.
We know that the polynomials $I\text{-}ii(\alpha, q)$ and $I\text{-}ii(r_{\mathbf{i}}(\alpha), q)$ are independent of the orientation of the quiver. Therefore we may assume that each vertex $(\mathbf{i}, r)$ of $\mathcal{Q}$ is a sink. It follows from Proposition 2.3.3 that the functors $S_{\mathbf{i}}^+$ and $S_{\mathbf{i}}^-$ are mutual inverses between the isomorphism classes of ii-indecomposables of $(\mathcal{Q}, \mathbf{a})$ of dimension vector $\beta = f^{-1}(\alpha)$ and those of $(s_{\mathbf{i}}\mathcal{Q}, \mathbf{a})$ of dimension vector $s_{\mathbf{i}}(\beta)$, all defined over $\mathbb{F}_q$.

N.B. Here we are writing $s_{\mathbf{i}}\mathcal{Q}$ for the quiver obtained from $\mathcal{Q}$ by inverting the orientation of each arrow with head $(\mathbf{i}, r)$. Thus each $(\mathbf{i}, r)$ is a source vertex of $s_{\mathbf{i}}\mathcal{Q}$.

In particular, we obtain that $I\text{-}ii(\alpha, q) = I\text{-}ii(r_{\mathbf{i}}(\alpha), q)$. Using Proposition 4.1.6 (and noting that $\overline{r_{\mathbf{i}}(\alpha)} = \bar{\alpha}$), we deduce that $A\text{-}ii(\alpha, q) = A\text{-}ii(r_{\mathbf{i}}(\alpha), q)$.
■

4.6 Subgroup and Quotient Group Actions

We now study the coefficients $b(m)_{(\mathbf{i},r)(\mathbf{j},s)}$ appearing in the power series $P_m(X, q)$ and show that they have a natural interpretation in terms of the quiver and automorphism.

We recall that the vertices of $\mathcal{Q}$ can be considered as pairs $(\mathbf{i}, r)$ for $\mathbf{i} \in \mathbf{I}$ and $r \in \mathbb{Z}/d_{\mathbf{i}}\mathbb{Z}$ such that $\mathbf{a}$ acts as $(\mathbf{i}, r) \mapsto (\mathbf{i}, r + 1)$.

Fix $m|n$ and consider the subgroup generated by $\mathbf{a}^m$ acting on $\mathcal{Q}$. This is obviously admissible and so we can consider the algebra $K\mathcal{Q}\#\langle\mathbf{a}^m\rangle$. As before, this will be Morita equivalent to the path algebra of another quiver, which we shall denote by $\mathcal{Q}(m)$. In particular, $\mathcal{Q}(1) = \widetilde{\mathcal{Q}}$ and $\mathcal{Q}(n) = \mathcal{Q}$.

The vertex set $\mathcal{I}(m)$ of $\mathcal{Q}(m)$ can be descibed as follows. Representatives for the vertex orbits of $\mathbf{a}^m$ acting $\mathcal{I}$ are $(\mathbf{i}, r)$ with $0 \leq r < (m, d_{\mathbf{i}})$, and each of these has orbit size $[m, d_{\mathbf{i}}]/m$. Therefore, by analogy with the construction in Chapter 3 (and noting that $\mathbf{a}^m$ has order n/m), we see that the elements of $\mathcal{I}(m)$ can be thought of as triples $(\mathbf{i}, r, \mu)$ with $0 \leq r < (m, d_{\mathbf{i}})$ and $0 \leq \mu < n/[m, d_{\mathbf{i}}]$.

Now consider an arrow $\rho : (\mathbf{i}, 0) \to (\mathbf{j}, l)$ and suppose that $\mathbf{a}^t(\rho) = \zeta^{tu}\rho$, where $t = [d_{\mathbf{i}}, d_{\mathbf{j}}]$. The simple $\Lambda_0\#\langle\mathbf{a}\rangle$-bimodule generated by ρ must split into (m, t) simple $\Lambda_0\#\langle\mathbf{a}^m\rangle$-bimodules with generators $\mathbf{a}^r(\rho)$ for $0 \leq r < (m, t)$. We note that $\mathbf{a}^r(\rho) : (\mathbf{i}, r) \to (\mathbf{j}, r + l)$ and $\mathbf{a}^{[m,t]}(\mathbf{a}^r(\rho)) = \zeta^{[m,t]u}\mathbf{a}^r(\rho)$. Using the primitive n/m-th root of unity ζ^m, we deduce that $\mathcal{Q}(m)$ has arrows $(\mathbf{i}, r, \mu) \to (\mathbf{j}, r + l, \nu)$ if and only if $\mu \equiv u + \nu \bmod n/[m, t]$.

N.B. We are considering $0 \leq r < (m, t)$ and writing $(\mathbf{i}, r, \mu)$ for $(\mathbf{i}, r \bmod (m, d_{\mathbf{i}}), \mu)$.

It follows that the quotient group $\langle\mathbf{a}\rangle/\langle\mathbf{a}^m\rangle$ acts naturally on $\mathcal{Q}(m)$ as admissible automorphisms, with the generator $\mathbf{a}_m := \mathbf{a} \bmod \langle\mathbf{a}^m\rangle$ acting on the vertices by $(\mathbf{i}, r, \mu) \mapsto (\mathbf{i}, r+1, \mu)$. The vertex orbits are therefore the pairs $(\mathbf{i}, \mu)$ with $\mathbf{i} \in \mathbf{I}$ and $0 \leq \mu < n/[m, d_{\mathbf{i}}]$ — which is precisely the set $\mathbf{I}(m)$ defined earlier.

N.B. This action should be contrasted with the action on $\mathcal{I}(m)$ of the dual group of $\langle\mathbf{a}^m\rangle$, which sends vertex $(\mathbf{i}, r, \mu)$ to $(\mathbf{i}, r, \mu+1)$.

Therefore we can construct a symmetric matrix $B(m)$ indexed by $\mathbf{I}(m)$. This is given by

$$b(m)_{(\mathbf{i},\mu)(\mathbf{j},\nu)} := \begin{cases} 2[m, d_{\mathbf{i}}]/m & \text{if } (\mathbf{i}, \mu) = (\mathbf{j}, \nu); \\ -\#\{\text{edges between orbits } (\mathbf{i}, \mu) \text{ and } (\mathbf{j}, \nu)\} & \text{if } (\mathbf{i}, \mu) \neq (\mathbf{j}, \nu). \end{cases} \tag{4.6.1}$$

Now, for $(\mathbf{i}, \mu) \neq (\mathbf{j}, \nu)$, we have

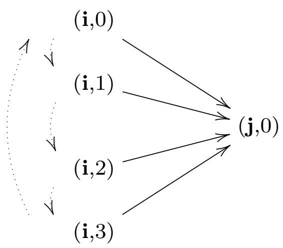
$$\sum_{r,s} \#\{\text{arrows } (\mathbf{i}, r, \mu) \to (\mathbf{j}, s, \nu)\} = (m, t) \sum_{\substack{\rho:(\mathbf{i},0)\to(\mathbf{j},l) \\ \mathbf{a}^t(\rho)=\zeta^{tu}\rho}} 1_{\{\mu \equiv u+\nu \bmod n/[m,t]\}}, \tag{4.6.2}$$

where we are again summing over generators for the simple submodules of $\Lambda_1\#\langle\mathbf{a}\rangle$. Hence the definitions for $b(m)$ given by (4.4.2) and (4.6.1) agree.

In particular, $(\mathcal{Q}(1), \mathbf{a}_1) = (\widetilde{\mathcal{Q}}, 1)$ and $B(1) = \widetilde{A}$, whereas $(\mathcal{Q}(n), \mathbf{a}_n) = (\mathcal{Q}, \mathbf{a})$ and $B(n) = B$.

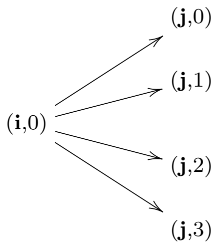
We illustrate this with an example.

Example 4.6.1. *Let $(\mathcal{Q}, \mathbf{a})$ be the quiver with automorphism of order 4*

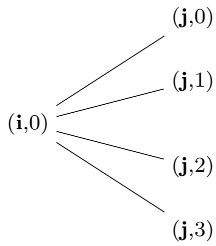


For $m = 1, 2, 4$ we describe the quiver with automorphism $(\mathcal{Q}(m), \mathbf{a}_m)$ and the *associated valued graph* $\Gamma(m)$.

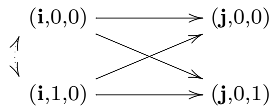
$(\mathcal{Q}(1), \mathbf{a}_1)$



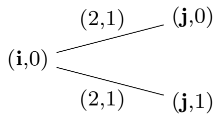
$\Gamma(1)$



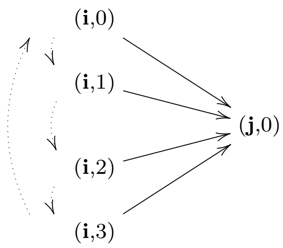
$(\mathcal{Q}(2), \mathbf{a}_2)$



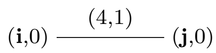
$\Gamma(2)$



$(\mathcal{Q}(4), \mathbf{a}_4)$



$\Gamma(4)$



Chapter 5

Number of Parameters

Let K be an algebraically closed field of charcteristic not dividing n. In this chapter we will show that the number of parameters $\mu\text{-}ii(\alpha, K)$ and the number of top-dimensional families of orbits $t\text{-}ii(\alpha, K)$ for the action of $\mathrm{GL}(f^{-1}(\alpha), K)$ on the constructible set $\text{Ind-ii}(\alpha, K)$ are given respectively by the degree and leading coefficient of the polynomial $A\text{-}ii(\alpha, q)$. In particular, these numbers are independent of the orientation of the quiver and the characteristic of the field, and are invariant under the action of the Weyl group.

The proof consists of two parts. We first show that the set $\text{Ind-ii}(\alpha, K)$ can be considered as the K-rational points of a constructible subset of an arithmetic scheme. Then we prove that the numbers $\mu\text{-}ii(\alpha, K)$ and $t\text{-}ii(\alpha, K)$ are constant for $\operatorname{char} K$ lying in a non-empty open subset of $\operatorname{Spec} \mathbb{Z}$ — in particular, for $\operatorname{char} K$ sufficiently large or 0.

Finally we employ the result of Lang-Weil on the number of rational points in a variety over a finite field, together with the results of the previous chapter, to show that for $K = \overline{\mathbb{F}}_q$ with $q \equiv 1 \bmod n$ the numbers $\mu\text{-}ii(\alpha, K)$ and $t\text{-}ii(\alpha, K)$ are given respectively by the degree and leading coefficient of the polynomial $A\text{-}ii(\alpha, q)$.

5.1 The set $V(\alpha, K)$

We recall from Section 2.4 that for $0 \leq \alpha \in \mathbb{Z}\mathbf{I}$, Rep-ii$(\alpha, K)$ (respectively Ind-ii(α, K)) is the constructible subset of the affine space $\mathrm{Rep}(f^{-1}(\alpha), K)$ corresponding to the set of all ii-representations (repsectively ii-indecomposables) of dimension vector $f^{-1}(\alpha)$.

We introduce the following set.

$$V(\alpha, K) := \{(g, X) \in \mathrm{GL}(f^{-1}(\alpha), K) \times \text{Ind-ii}(\alpha, K) \mid g \cdot X = X\}. \tag{5.1.1}$$

This is a closed subset of the constructible subset $\mathrm{GL}(f^{-1}(\alpha), K) \times \text{Ind-ii}(\alpha, K)$, so is itself constructible.

Proposition 5.1.1. *We have*

$$\dim V(\alpha, K) - \dim \mathrm{GL}(f^{-1}(\alpha), K) = \dim_{\mathrm{GL}(f^{-1}(\alpha),K)} \text{Ind-ii}(\alpha, K) = \mu\text{-}ii(\alpha, K)$$

and

$$\operatorname{top} V(\alpha, K) = \operatorname{top}_{\mathrm{GL}(f^{-1}(\alpha),K)} \text{Ind-ii}(\alpha, K) = t\text{-}ii(\alpha, K).$$

Proof.
Let $V'(\alpha, K) = \{(g, X) \in \mathrm{GL}(f^{-1}(\alpha), K) \times \mathrm{Rep}(f^{-1}(\alpha), K) \mid g \cdot X = X\}$ and denote by π the projection $V'(\alpha, K) \twoheadrightarrow \mathrm{Rep}(f^{-1}(\alpha), K)$. Then the fibre over a point $X \in \mathrm{Rep}(f^{-1}(\alpha), K)$ is

$$\pi^{-1}(X) = \{(g, X) \mid g \cdot X = X\} \cong \mathrm{Aut}(X).$$

In particular, since this is open in the affine space $\mathrm{End}(X)$, it is irreducible.

The result now follows immediately from Proposition B.5.
■

5.2 Arithmetic Schemes

In this section we show that the set $V(\alpha, K)$ can be viewed as the K-rational points of a constructible subset of an arithmetic scheme — that is, a scheme of finite type over $\mathbb{Z}$ (see [10]). Since we shall only be concerned with affine schemes, this is equivalent to the ring representing the scheme being finitely generated over $\mathbb{Z}$. In particular, all our schemes will be noetherian and all morphisms will be of finite type, so we can apply Chevalley's Theorem, Theorem A.4.

We know that the category of affine schemes is equivalent to the category of rings with arrows reversed (see Section A.1 of the Appendix). Alternatively, we can characterise schemes as representable functors. This follows from Yoneda's Lemma:

Theorem 5.2.1. *Let X and Y be contravariant functors from rings to sets represented by the rings A and B. Then the natural maps $X \to Y$ correspond to ring homomorphisms $B \to A$.*

We also have the following theorem, which determines when a functor is representable. (See [10] or [41].)

Theorem 5.2.2. *Let X be a functor from rings to sets. Then X is representable if and only if there is some family of equations over $\mathbb{Z}$ such that the set $X(R)$ is precisely the set of solutions in R.*

Consider the functors

$$\mathrm{Rep}(\alpha) : R \to \prod_{\rho} \mathbb{M}\left(\alpha_{t(\rho)} \times \alpha_{s(\rho)}, R\right) \qquad \text{and} \qquad \mathrm{GL}(\alpha) : R \to \prod_{i} \mathrm{GL}(\alpha_i, R). \tag{5.2.1}$$

These are clearly representable, say represented respectively by the algebras A and B. In fact, $\mathrm{Rep}(\alpha)$ is represented by a polynomial ring $A = \mathbb{Z}[\{T(\rho)_{ij}\}]$, with variables indexed by the positions in the matrices.

Furthermore, $\mathrm{GL}(\alpha)$ is actually an affine group scheme and, since $\mathrm{GL}(\alpha, R)$ acts naturally on $\mathrm{Rep}(\alpha, R)$, we obtain a morphism $\mathrm{GL}(\alpha) \times \mathrm{Rep}(\alpha) \to \mathrm{Rep}(\alpha)$, say corresponding to the ring homorphism $\phi : A \to B \otimes A$.

We also see that for $\alpha \in (\mathbb{Z}\mathcal{I})^{\langle \mathbf{a} \rangle}$, the automorphism $\mathbf{a}$ acts naturally on $\mathrm{Rep}(\alpha)$. This corresponds to the algebra automorphism ψ of A given by $T(\rho)_{ij} \mapsto T(\mathbf{a}(\rho))_{ij}$ (up to multiplication by a root of unity, see (2.3.5)).

It follows that for any $0 \leq \alpha \in \mathbb{Z}\mathbf{I}$ there is an affine scheme $M(\alpha)$ such that

$$M(\alpha, R) = \{(g, X) \in \mathrm{GL}(f^{-1}(\alpha), R) \times \mathrm{Rep}(f^{-1}(\alpha), R) \mid g \cdot X = {}^{\mathfrak{a}}X\}. \quad (5.2.2)$$

The algebra representing $M(\alpha)$ can be given explicitly as $B \otimes A/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by all elements of the form $\phi(a) - 1 \otimes \psi(a)$ for $a \in A$.

The projection map $M(\alpha) \to \mathrm{Rep}(f^{-1}(\alpha))$ is a morphism and so by Chevalley's Theorem its set-theoretic image Rep-ii(α) is constructible. We now show that for K an algebraically closed field, the set of K-valued points of Rep-ii(α) is precisely the set Rep-ii(α, K).

Proposition 5.2.3. *Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. Then for K an algebraically closed field we have $f((X(K)) = (f(X))(K)$. That is, the images of the K-valued points of X are precisely the K-valued points of the constructible subset $f(X) \subset Y$.*

Proof.
Let $\mathcal{O}_X$ be the sheaf of rings on X. For any point $x \in X$, we denote the maximal ideal and the residue field of the local ring $\mathcal{O}_{X,x}$ by $\mathfrak{m}_{X,x}$ and $\kappa(x)$ respectively. A point x of X is K-valued if there exists a non-zero homomorphism $\kappa(x) \to K$, or equivalently if x is the image of a morphism $\operatorname{Spec} K \to X$.

For any morphism $f : X \to Y$ we have a morphism of sheaves on Y, $\psi : \mathcal{O}_Y \to f_*\mathcal{O}_X$. Here $f_*\mathcal{O}_X$ is the direct image sheaf, given by $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$. Consequently, for each $x \in X$ we get a local homomorphism of local rings $\psi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ and hence a homomorphism $\kappa(f(x)) \to \kappa(x)$.

Clearly we have the inclusion $f(X(K)) \subset (f(X))(K)$. On the other hand, if $y \in Y$ then the fibre $f^{-1}(y)$ is homeomorphic to the fibred product $X \times_Y \operatorname{Spec} \kappa(y)$ (by Lemma A.1). Take any closed point $z \in X \times_Y \operatorname{Spec} \kappa(y)$ and let x be its image in X. Then we have homomorphisms $\kappa(y) \to \kappa(x) \to \kappa(z)$, and since $X \times_Y \operatorname{Spec} \kappa(y)$ is a scheme of finite type over $\kappa(y)$, $\kappa(z)$ is algebraic over $\kappa(y)$. Therefore $\kappa(x)$ is also algebraic over $\kappa(y)$.

In particular, if K is algebraically closed and $y \in f(X)$ is a K-valued point, then there exists a K-valued point of X mapping to y.

■

For any proper decomposition $\alpha = \beta + \gamma$ and any ring R, consider the morphism

$$\theta_{\beta,\gamma}(R) : \mathrm{GL}(f^{-1}(\alpha), R) \times \mathrm{Rep}(f^{-1}(\beta), R) \times \mathrm{Rep}(f^{-1}(\gamma), R) \to \mathrm{Rep}(f^{-1}(\alpha), R),$$
$$(g, X, Y) \mapsto g \cdot (X \oplus Y). \tag{5.2.3}$$

Since $\theta_{\beta,\gamma}$ behaves naturally with respect to ring homomorphisms, we again obtain a morphism of schemes

$$\theta_{\beta,\gamma} : \mathrm{GL}(f^{-1}(\alpha)) \times \mathrm{Rep}(f^{-1}(\beta)) \times \mathrm{Rep}(f^{-1}(\gamma)) \to \mathrm{Rep}(f^{-1}(\alpha)). \tag{5.2.4}$$

In particular, its image is constructible. Furthermore, we can consider the image under $\theta_{\beta,\gamma}$ of the set $\mathrm{GL}(f^{-1}(\alpha)) \times \text{Rep-ii}(\beta) \times \text{Rep-ii}(\gamma)$. This is again a constructible set and so the complement in Rep-ii(α) of the union of all such images, which we shall denote by Ind-ii(α), is constructible. We can now apply Proposition 5.2.3 to deduce that over any algebraically closed field K, the K-valued points of Ind-ii(α) are precisely the ii-indecomposables of dimension vector $f^{-1}(\alpha)$ over K.

Finally, we have an affine arithmetic scheme $V'(\alpha) \subset \mathrm{GL}(f^{-1}(\alpha)) \times \mathrm{Rep}(f^{-1}(\alpha))$ such that

$$V'(\alpha, R) = \{(g, X) \in \mathrm{GL}(f^{-1}(\alpha), R) \times \mathrm{Rep}(f^{-1}(\alpha), R) \mid g \cdot X = X\}. \tag{5.2.5}$$

The projection $\pi : V'(\alpha) \to \mathrm{Rep}(f^{-1}(\alpha))$ is a surjective morphism and so we can define a constructible set $V(\alpha)$ by

$$V(\alpha) := \pi^{-1}(\text{Ind-ii}(\alpha)). \tag{5.2.6}$$

We deduce that for any algebraically closed field K, the set of K-valued points of $V(\alpha)$ is precisely the set $V(\alpha, K)$ defined in the previous section.

5.3 Changing Characteristic

We now apply some scheme-theoretic results of Grothendieck to prove

Theorem 5.3.1. *The numbers* $\dim V(\alpha, K)$ *and* $\operatorname{top} V(\alpha, K)$ *are constant for* K *an algebraically closed field of characteristic 0 or* $p \gg 0$*. As a consequence, the same is true of the numbers* $\mu\text{-}ii(\alpha, K)$ *and* $t\text{-}ii(\alpha, K)$*.*

We first show that these numbers depend only on the characteristic of the field and not on the particular algebraically closed field.

Theorem 5.3.2. *Let* K *be a field and* X *an affine algebraic* K*-scheme. Then there exists a finite field extension* L/K *such that every irreducible component of* $(X_L)_{\mathrm{red}}$ *is geometrically integral, where* $X_L := X \otimes_K L$*.*

Proof.
This is a special case of [14], Corollary 4.5.10 and Proposition 4.6.6. See also Theorem A.10.
■

Corollary 5.3.3. *Let* K *be algebraically closed,* X *an affine algebraic* K*-scheme and* $Z \subset X$ *a constructible subset. Then for any field extension* L/K*, and writing* $f : X_L \to X$ *for the pull-back morphism, we have*

$$\dim f^{-1}(Z) = \dim Z \qquad \textit{and} \qquad \operatorname{top} f^{-1}(Z) = \operatorname{top} Z.$$

Proof.
Expressing $Z = Z_1 \cup \cdots \cup Z_m$ as the union of its irreducible components, it is enough to prove that each $f^{-1}(Z_i)$ is irreducible and has the same dimension as Z_i. Therefore we may assume that Z is irreducible.

Set $W := \overline{Z}$ and give it the reduced subscheme structure. Then W is an integral affine algebraic K-scheme and, since K is algebraically closed, the theorem implies that W is geometrically integral.

Let $W = \operatorname{Spec} A$, where A is a finitely generated K-domain. The Noether Normalisation Lemma says that there exists a polynomial subalgebra $K[T_1, \dots, T_d]$ of A over which A is finite. N.B. d is then the dimension of W. Now $A \otimes_K L$ is finite over $L[T_1, \dots, T_d]$ and so W_L has dimension d.

Finally, f is an open map (Lemma A.6) and so $f^{-1}(Z)$ is dense in $f^{-1}(W) = W_L$. Therefore $f^{-1}(Z)$ is irreducible of dimension $\dim Z$.

■

In particular, we can consider $\dim V(\alpha, K)$ and $\operatorname{top} V(\alpha, K)$ for any algebraically closed field K and these numbers are constant within each characteristic.

We now consider the following situation. Let X and S be affine noetherian schemes and $f : X \to S$ a dominant morphism of finite type. We also assume that S is integral with generic point η. For $s \in S$ we write $X_s := X \times_S \operatorname{Spec} \kappa(s)$, which is homeomorphic to $f^{-1}(s)$.

Theorem 5.3.4. *Let $f : X \to S$ be as above. Then there exists an open neighbourhood of η over which the function $s \mapsto \dim X_s$ is constant. Furthermore, if X_η is geometrically integral, then we may assume the same is true for each X_s.*

Proof.
This is a special case of [15], Corollary 9.2.6.2 and Theorem 9.7.7. See also Theorems A.14 and A.16.

■

Now suppose that Z is a constructible subset of X. For $s \in S$ and $L/\kappa(s)$ any field extension we have the pull-back morphism $g_L : X_L := X_s \otimes_{\kappa(s)} L \to X$. We shall denote the constructible subset $g_L^{-1}(Z)$ by Z_L. In particular, we have the sets $Z_s := Z_{\kappa(s)}$ and $Z_{\bar{s}} := Z_{\overline{\kappa(s)}}$.

Corollary 5.3.5. *Let $f : X \to S$ be as above and let $Z \subset X$ be constructible with $Z_\eta := Z \cap X_\eta$ non-empty. Then there exists an open neighbourhood of η over which the numbers $\dim Z_{\bar{s}}$ and $\operatorname{top} Z_{\bar{s}}$ are constant.*

Proof.
Replacing X with the closure of Z, we may assume that Z is dense in X. We

know (Lemma A.2) that the maximal points of X_η are all maximal in X and since Z is dense in X, it contains all the maximal points of X. In particular, Z_η is dense in X_η and so there exists an open neighbourhood of η over which Z_s is dense in X_s (Proposition A.13). Also, since the map $X_{\bar{s}} \to X_s$ is open, we know that $Z_{\bar{s}}$ is dense in $X_{\bar{s}}$ and hence it is enough to prove the result for X itself.

Let $K = \kappa(\eta)$. Then X_η is an affine algebraic K-scheme and so by Theorem 5.3.2 there exists a finite extension L/K such that every irreducible component of $(X_L)_{\text{red}}$ is geometrically integral, where $X_L := X_\eta \otimes_K L$.

We can write $S = \operatorname{Spec} A$, where A is a finitely generated domain with fraction field K. Let B be the integral closure of A in L and set $T = \operatorname{Spec} B$. Then T is again an affine noetherian integral scheme, with generic point ξ such that $\kappa(\xi) = L$. Let $Y = X \times_S T$ and write $g : Y \to T$ for the pull-back morphism.

Let $Y_1, \dots, Y_m$ be the closures in Y of the irreducible components of Y_ξ. We give them the reduced subscheme structures, so that the $(Y_i)_\xi$ are all geometrically integral. Let $W_i = Y_i - \bigcup_{j \neq i} Y_j$, which is open in Y_i, and $Y' = Y - \bigcup_i Y_i$. We note that $Y'_\xi = \emptyset$ and each $(W_i)_\xi \neq \emptyset$. Therefore, by Chevalley's Theorem, there exists an open neighbourhood of ξ in T over which $Y'_t = \emptyset$ and each $(W_i)_t \neq \emptyset$. Moreover, by Theorem 5.3.4, we may also assume that each $(Y_i)_t$ is geometrically integral of dimension $\dim(Y_i)_\xi$. In particular, the irreducible components of Y_t are precisely the $(Y_i)_t$.

Thus $\dim Y_t = \dim Y_\xi$ and $\operatorname{top} Y_t = \operatorname{top} Y_\xi$ for all t in an open neighbourhood U of ξ. Furthermore, since each $(Y_i)_t$ is geometrically integral, we have $\dim Y_{\bar{t}} = \dim Y_t$ and $\operatorname{top} Y_{\bar{t}} = \operatorname{top} Y_t$.

Finally, since the map $T \to S$ is finite and dominant, the image of U contains an open neighbourhood of η. Also, if $t \mapsto s$, then $\kappa(t)/\kappa(s)$ is a finite extension. In particular $\overline{\kappa(t)} = \overline{\kappa(s)}$. Therefore

$$Y_{\bar{t}} = (Y \times_T \operatorname{Spec} \kappa(t)) \otimes_{\kappa(t)} \overline{\kappa(t)} = (X \times_S \operatorname{Spec} \kappa(s)) \otimes_{\kappa(s)} \overline{\kappa(s)} = X_{\bar{s}}$$

and the result follows.

■

Now consider the arithmetic scheme $V'(\alpha)$ and the constructible subset $V(\alpha)$. If K is an algebraically closed field of characteristic $p \geq 0$, then we can consider the constructible subset $V(\alpha)_{\bar{p}}$ of the affine algebraic scheme $V'(\alpha)_{\bar{p}}$ and the set of all K-valued points equals

$$V(\alpha)(K) = V(\alpha)_{\bar{p}}(K) = V(\alpha, K). \tag{5.3.1}$$

Therefore by Corollary 5.3.3 the numbers $\dim V(\alpha, K)$ and $\operatorname{top} V(\alpha, K)$ are constant within each characteristic.

Moreover, if $V(\alpha)_0$ is non-empty, then by Corollary 5.3.5 these numbers are constant on an open neighbourhood of 0. Finally, if $V(\alpha)_0$ is empty, then its image in $\operatorname{Spec} \mathbb{Z}$ must be contained in a proper closed subset and so $V(\alpha)_p$ is empty for almost all primes p.

This proves Theorem 5.3.1.

5.4 The Lang-Weil Theorem

We now prove

Theorem 5.4.1. *The numbers μ-ii(α, K) and t-ii(α, K) for K an algebraically closed field of characteristic 0 or $p > 0$ coprime to n are given respectively by the degree and leading coefficient of the polynomial A-ii(α, q).*

We note that Corollary 1.6.3 follows immediately from this and Theorem 1.6.2.

By Proposition 5.1.1, it is enough to consider the numbers $\dim V(\alpha, K)$ and $\operatorname{top} V(\alpha, K)$, and by Theorem 5.3.1 we can restrict ourselves to the case $K = \overline{\mathbb{F}}_q$ for $q \equiv 1 \bmod n$. We quote the following theorem.

Theorem 5.4.2 (Lang-Weil). *Let V be an irreducible variety of dimension d defined over $\mathbb{F}_q$. Then*

$$|V(\mathbb{F}_{q^r})| = q^{rd} + \mathcal{O}\big(q^{r(d-\frac{1}{2})}\big) \quad \textit{as } r \to \infty.$$

Proof.
The original proof can be found in [29] but see also [38], Chapter 7.
■

Corollary 5.4.3. *Let V be a variety defined over $\mathbb{F}_q$ and set $d = \dim V$ and $t = \operatorname{top} V$. Then for some integer s*

$$|V(\mathbb{F}_{q^{sr}})| = tq^{srd} + \mathcal{O}\big(q^{sr(d-\frac{1}{2})}\big) \quad \textit{as } r \to \infty.$$

Proof.
Let $V = V_1 \cup \cdots \cup V_m$ be the decomposition of V into its irreducible components. Let $V_1, \ldots, V_t$ have dimension d and $V_{t+1}, \ldots, V_m$ have dimension $< d$. For some s, each of the V_i is defined over $\mathbb{F}_{q^s}$ and so $U_i := V_i - \bigcup_{j \neq i} V_j$ is also defined over $\mathbb{F}_{q^s}$. Let $W_1 = U_1 \dot\cup \cdots \dot\cup U_t$ and $W_2 = V - W_1$.

By the theorem, $|W_1(\mathbb{F}_{q^{sr}})| = tq^{srd} + \mathcal{O}\big(q^{sr(d-\frac{1}{2})}\big)$ as $r \to \infty$. Also, W_2 has dimension strictly smaller than d, so $|W_2(\mathbb{F}_{q^{sr}})| = \mathcal{O}\big(q^{sr(d-1)}\big)$ as $r \to \infty$ by induction on d. Since $|V(\mathbb{F}_{q^{sr}})| = |W_1(\mathbb{F}_{q^{sr}})| + |W_2(\mathbb{F}_{q^{sr}})|$, the result follows.
■

Corollary 5.4.4. *Let V be an affine variety defined over $\mathbb{F}_q$ and $Z \subset V$ a constructible subset. Assume that Z is also defined over $\mathbb{F}_q$ — that is, Z is given by the vanishing or non-vanishing of polynomials defined over $\mathbb{F}_q$. If $d = \dim Z$ and $t = \operatorname{top} Z$, then there exists an integer s such that*

$$|Z(\mathbb{F}_{q^{sr}})| = tq^{srd} + \mathcal{O}\big(q^{sr(d-\frac{1}{2})}\big) \quad \textit{as } r \to \infty.$$

Proof.
Let V have co-ordinate ring A, a finitely generated $\mathbb{F}_q$-algebra. Since Z is defined over $\mathbb{F}_q$, it can be viewed as a constructible set in $\operatorname{Spec} A$. The closure W of Z in V corresponds to the closure of Z in $\operatorname{Spec} A$, so is defined over $\mathbb{F}_q$. Similarly, we can find an open dense subset U of W in Z, also defined over $\mathbb{F}_q$.

We can now apply Corollary 5.4.3 to U and W to deduce that, for some integer s, the numbers $|U(\mathbb{F}_{q^{sr}})|$ and $|W(\mathbb{F}_{q^{sr}})|$ are both of the form $tq^{srd} + \mathcal{O}\big(q^{sr(d-\frac{1}{2})}\big)$

as $r \to \infty$. Since

$$|U(\mathbb{F}_{q^{sr}})| \leq |Z(\mathbb{F}_{q^{sr}})| \leq |W(\mathbb{F}_{q^{sr}})|,$$

the result follows.

■

In particular, we can apply the result to $V(\alpha, K)$ for K algebraically closed of characteristic $p > 0$ coprime to n. In this case, we get a point $(g, X) \in V(\alpha, \mathbb{F}_q)$ if and only if $g \in \mathrm{GL}(f^{-1}(\alpha), q)$ and $X \in \mathrm{Rep}(f^{-1}(\alpha), q)$ such that $K \otimes X$ is an ii-indecomposable and $g \cdot X = X$. However, ${}^{\mathbf{a}}X$ is also defined over $\mathbb{F}_q$ and so by Lemma 4.1.1 $X \cong {}^{\mathbf{a}}X$. That is, X is an absolutely ii-indecomposable. Thus

$$V(\alpha, \mathbb{F}_q) = \{(g, X) \mid X \text{ absolutely ii-indecomposable and } g \cdot X = X\}. \quad (5.4.1)$$

Therefore, for $q \equiv 1 \bmod n$,

$$|V(\alpha, \mathbb{F}_q)| = |\,\mathrm{GL}(f^{-1}(\alpha), \mathbb{F}_q)| A\text{-}ii(\alpha, q) \quad (5.4.2)$$

is a polynomial of degree $\dim \mathrm{GL}(f^{-1}(\alpha), K) + \deg A\text{-}ii(\alpha, q)$ and with leading coefficient that of $A\text{-}ii(\alpha, q)$. Therefore

$$\mu\text{-}ii(\alpha, K) = \dim V(\alpha, K) - \dim \mathrm{GL}(\alpha, K) = \deg A\text{-}ii(\alpha, q) \quad (5.4.3)$$

and

$$t\text{-}ii(\alpha, K) = \mathrm{top}\, V(\alpha, K) = \text{leading coefficient of } A\text{-}ii(\alpha, K). \quad (5.4.4)$$

This completes the proof of Theorem 5.4.1.

Chapter 6

The Affine Quivers

In this section, we shall study the affine quivers and all their possible admissible automorphisms. We first recall some properties of affine (valued) graphs.

Let Γ be a connected valued graph. Then Γ is affine if and only if there exists $\delta_\Gamma > 0$ such that $(\delta_\Gamma, -)_\Gamma \equiv 0$, and this δ_Γ is unique up to a scalar ([25], Proposition 4.7). We can take δ_Γ to be integral such that $\bar{\delta}_\Gamma = 1$ and then $\Delta(\Gamma)^{\text{im}}_{+} = \{m\delta_\Gamma \mid m \in \mathbb{N}\}$.

We also recall the tier number associated to Γ. This is the smallest positive number t for which

$$\Delta(\Gamma)^{\text{re}} + t\delta_\Gamma = \Delta(\Gamma)^{\text{re}}.$$

We note that t equals 1, 2 or 3 for all affine valued graphs Γ and that $t = 1$ for all unvalued graphs (or symmetric GCMs).

If we assume that $\mathcal{Q}$ is a connected affine quiver and $\mathbf{a}$ an admissible automorphism, then $\mathbf{a}(\delta_\mathcal{Q}) = \delta_\mathcal{Q}$. Similarly, using Lemma 2.1.2, we see that if Γ is the associated valued graph, then Γ is affine with $\delta_\Gamma = f(\delta_\mathcal{Q})$. We shall write δ for both $\delta_\mathcal{Q}$ and δ_Γ.

On the other hand, consider the dual quiver with automorphism $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$. This has as its associated valued graph $\widetilde{\Gamma}$, the dual of Γ. Hence $\widetilde{\Gamma}$ is also affine and so by Lemma 2.1.2, $\widetilde{\mathcal{Q}}$ is a disjoint union of copies of one affine quiver.

Our aim will be to describe the action of $\mathbf{a}$ on the isomorphism classes of indecomposables of $\mathcal{Q}$, and in particular to compute the polynomials $A\text{-}ii(m\delta, q)$. These are independent of the orientation, so we may assume that $\mathcal{Q}$ has no oriented cycles and hence $K\mathcal{Q}$ is finite dimensional.

6.1 Classification of Indecomposables

Let $\mathcal{Q}$ be a connected affine quiver without oriented cycles. There is an explicit description of the set of isomorphism classes of indecomposable representations of $\mathcal{Q}$. For K algebraically closed, this was obtained independently by Nazarova [34] and by Donovan and Frieslich [9], but see also [8, 37].

This classification divides the indecomposables into three sets — the preprojective, preinjective and regular indecomposables. These can be described as follows.

The Euler form (or Ringel form) of $\mathcal{Q}$ is defined by

$$\langle \alpha, \beta \rangle := \sum_{i \in \mathcal{I}} \alpha_i \beta_i - \sum_{\rho : i \to j} \alpha_i \beta_j. \tag{6.1.1}$$

This is a bilinear form on $\mathbb{Z}\mathcal{I}$, dependent on the orientation of $\mathcal{Q}$. We note that this is related to the symmetric bilinear form defined earlier by

$$\langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle = (\alpha, \beta).$$

The defect of an indecomposable representation X is now given by

$$\operatorname{defect}(X) := \langle \delta, \underline{\dim}\, X \rangle, \tag{6.1.2}$$

where δ denotes the minimal positive imaginary root.

We let τ denote the Auslander-Reiten translate [1, 37]. Then for X indecomposable we have

1. X is preprojective if and only if $\tau^r X = 0$ for $r \gg 0$, which is if and only if $\operatorname{defect}(X) < 0$;

2. X is preinjective if and only if $\tau^r X = 0$ for $r \ll 0$, which is if and only if $\text{defect}(X) > 0$;
3. X is regular if and only if $\tau^{-r}\tau^r X \cong X$ for all $r \in \mathbb{Z}$, which is if and only if $\text{defect}(X) = 0$.

It follows that all non-regular indecomposables have dimension vectors that are real roots. Therefore we shall mostly be interested in the regular indecomposables.

We call a representation regular if it is a direct sum of regular indecomposables. The full subcategory of regular representations is abelian and serial — that is, each regular indecomposable has a unique composition series with regular simple factors. In particular, each regular indecomposable is completely determined by its regular top and regular length. If E is a regular simple and r a positive integer, we write $E[r]$ for the regular indecomposable with regular top E and regular length r.

In fact, every regular simple E has finite period under τ and the regular composition factors of $E[r]$ are (from the top) $E, \tau E, \ldots, \tau^{r-1}E$. Moreover, the component of the Auslander-Reiten quiver containing E is a tube, consisting of all the indecomposables of the form $\tau^i E[r]$.

The set of tubes is parametrised by the projective line $\mathbb{P}^1 K$ and almost all are homogeneous (have period 1). The type of this family lists the periods greater than 1. This type is independent of the orientation in all cases except $\widetilde{\mathbb{A}}_l$, in which case we have type (a, b), where a and b are the numbers of arrows pointing clockwise and anticlockwise respectively.

6.2 Automorphisms of Affine Quivers

We are now in a position to examine how $\mathbf{a}$ acts on the set of isomorphism classes of indecomposables, and hence describe the set of ii-indecomposables.

To this end, we first note that $\mathbf{a}\tau$ and $\tau\mathbf{a}$ are naturally isomorphic functors on $\text{Rep}(\mathcal{Q})$. (This holds for all artin algebras, as noted in the proof of Lemma 4.1 of

[36]). Therefore the actions of $\mathbf{a}$ and τ are both determined up to isomorphism by how they act on the positive real roots $\Delta(\mathcal{Q})_+^{\mathrm{re}}$ and on the regular simples.

In particular, $\mathbf{a}$ fixes the isomorphism classes of the preprojective, preinjective and regular indecomposables set-wise, and if E is a regular simple, then ${}^{\mathbf{a}}(E[r]) \cong ({}^{\mathbf{a}}E)[r]$.

We now carry out a case-by-case analysis for the automorphisms of the affine quivers. We fix a finite base field $\mathbb{F}_q$ containing a primitive n-th root of unity ζ and denote the algebraic closure of $\mathbb{F}_q$ by K.

We observe the following. Let $\mathbf{a}$ be an automorphism of $K\mathcal{Q}$ and let $\rho : i \to j$ be an arrow such that $\mathbf{a}^t(\rho) = \mu\rho$, where $t = [d_{\mathbf{i}}, d_{\mathbf{j}}]$ and $\mu^{n/t} = 1$. Consider the inner automorphism ϕ of $K\mathcal{Q}$ given by conjugation by $1 + (\nu - 1)\sum_{j\in\mathbf{j}} \varepsilon_j$, for some n-th root of unity ν. This is a graded automorphism of $K\mathcal{Q}$ which acts by multiplying each arrow $\sigma : k \to j$ by ν^{-1}, multiplying each arrow $\sigma : j \to k$ by ν and leaving everything else fixed.

Therefore, the admissible automorphism $\mathbf{a}' := \mathbf{a}\phi$ satisfies $(\mathbf{a}')^t(\rho) = \nu^{-t}\mu\rho$. In particular, if $\mathcal{Q}$ is a tree, then up to inner automorphisms of the form ϕ, we may assume that $\mathbf{a}^t(\rho) = \rho$ for all arrows ρ. On the other hand, if $\mathcal{Q}$ is of type $\widetilde{\mathbb{A}}_l$, then we may assume that $\mathbf{a}^t(\rho) = \rho$ for all but one arrow ρ.

We note that these inner automorphisms do not affect the number of isomorphism classes of ii-indecomposables.

In each of the following cases, we follow the same procedure. We first describe the regular simples for the tubes of periods $p > 1$. Since these will have dimension vectors that are real roots, it is enough to write down these roots. For example, if there is a tube of period p, then we shall write down the roots $\alpha_1, \dots, \alpha_p$ such that $\tau V(\alpha_i) \cong V(\alpha_{i+1})$, where $V(\alpha_i)$ is the regular simple with dimension vector α_i.

We also list the homogeneous regular simples $T(\lambda)$ for λ lying in some subset of $\mathbb{P}^1 K$. Note that each $T(\lambda)$ has dimension vector δ.

We then determine how $\mathbf{a}$ acts on each of the regular simples, and hence on each of the tubes. Since $\mathbf{a}$ preserves the sets of preprojective and preinjective indecomposables, and these have negative or positive defect resepctively, only the regular indecomposables give rise to ii-indecomposables of dimension vector a multiple of δ. Therefore this will be enough to calculate the numbers $A\text{-}ii(m\delta, q)$.

All that remains is to work out how many isomorphism classes of ii-indecomposables are defined over $\mathbb{F}_q$. We have that each regular simple $V(\alpha_i)$ is defined over $\mathbb{F}_q$ by Kac's Theorem, since it has dimension vector a real root. On the other hand, if we have an ii-indecomposable of the form

$$T(\lambda_1) \oplus T(\lambda_2) \oplus \cdots \oplus T(\lambda_r),$$

then this is defined over $\mathbb{F}_q$ if and only if it is fixed by the Frobenius automorphism acting on $\mathrm{Rep}(\mathcal{Q})$. This is equivalent to the set $\{\lambda_1, \ldots, \lambda_r\}$ being fixed by the Frobenius map $x \mapsto x^q$, or saying the the λ_i are the roots of some polynomial defined over $\mathbb{F}_q$.

N.B. The notation we use for the affine quivers is taken from the paper by Dlab and Ringel [8], except we denote the Kronecker graph $\widetilde{\mathbb{A}}_{11}$ by $\widetilde{\mathbb{A}}_1$ and the valued graph $\cdot \xrightarrow{(4,1)} \cdot$ by $\widetilde{\mathbb{A}}_{11}$. We always denote the rank of the associated symmetrisable GCM by l, so Γ has $l+1$ vertices. Also, as far as possible, we order the cases into pairs such that Case $2m$ has valued graph Γ with tier number 1 and Case $2m+1$ has valued graph $\widetilde{\Gamma}$, the dual of Γ.

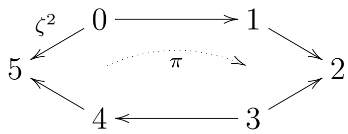
6.2.1 Case-by-case Analysis

Case 1.

Let $\mathcal{Q}$ be the graph $\widetilde{\mathbb{A}}_{t(l+1)-1}$ with vertices labelled clockwise from 0 to $t(l+1)-1$ and with the orientation $r(l+1) \to r(l+1)-1$ for $0 \leq r < t$ and all other arrows clockwise. Let ρ be the arrow $0 \to t(l+1)-1$. We define an admissible automorphism $\mathbf{a}$ on $\mathcal{Q}$ such that $i \mapsto (l+1)+i$ and $\mathbf{a}^t(\rho) = \zeta^t(\rho)$. Then Γ is the graph $\widetilde{\mathbb{A}}_l$, which has tier number 1.

N.B. The dual quiver with automorphism is again of this type, with t everywhere replaced by n/t.

For example, if $l = 2$ and $t = 2$, then $(\mathcal{Q}, \mathbf{a})$ is



The type of the family of tubes for $\mathcal{Q}$ is (tl, t) and the dimension vectors of the non-homogeneous regular simples are

$$\alpha_{rl} := e_{r(l+1)-1} + e_{r(l+1)}, \quad \alpha_{rl+i} := e_{r(l+1)+i} \quad \text{for } 1 \leq i < l \text{ and } 0 \leq r < t$$

and

$$\beta_r := e_{(t-r)(l+1)-1} + \cdots + e_{(t-r-1)(l+1)} \quad \text{for } 0 \leq r < t.$$

The homogeneous regular simples $T(\lambda)$ are parametrised by $0 \neq \lambda \in K$. These have a copy of the field K at each vertex and each arrow acts as the identity except for the arrow ρ, which is multiplication by λ.

We have that $\mathbf{a}(\alpha_{rl+i}) = \alpha_{(r+1)l+i}$ and $\mathbf{a}(\beta_r) = \beta_{r-1}$, which determines the action of $\mathbf{a}$ on these two tubes. Also, ${}^{\mathbf{a}}T(\lambda) \cong T(\zeta^{-t}\lambda)$. Therefore we obtain ii-indecomposables

$$T(\lambda) \oplus T(\zeta^t\lambda) \oplus \cdots \oplus T(\zeta^{n-t}\lambda) \quad \lambda \in K - \{0\},$$
$$V(\beta_0) \oplus \cdots \oplus V(\beta_{t-1}) \qquad \text{and} \qquad V(\alpha_i) \oplus \cdots \oplus V(\alpha_{(t-1)l+i}) \quad 0 \leq i < l.$$

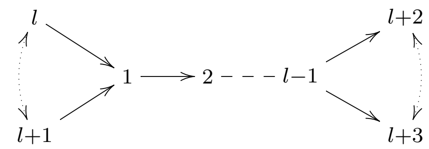
Now $T(\lambda) \oplus \cdots \oplus T(\zeta^{n-t}\lambda)$ is defined over $\mathbb{F}_q$ if and only if the scalars $\lambda, \zeta^t\lambda, \ldots, \zeta^{n-t}\lambda$ are the roots of a polynomial defined over $\mathbb{F}_q$, which is if and only if $\lambda^{n/t} \in \mathbb{F}_q$.

Hence

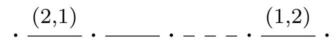
$$A\text{-}ii(m\delta, q) = \begin{cases} l+1 & \text{if } m \not\equiv 0 \bmod n/t; \\ q+l & \text{if } m \equiv 0 \bmod n/t. \end{cases}$$

Case 2.

Let $(\mathcal{Q}, \mathbf{a})$ be given by the quiver $\widetilde{\mathbb{D}}_{l+2}$ with $\mathbf{a}$ of order 2 as shown.



Then Γ is the quiver $\widetilde{\mathbb{C}}_l$, of tier number 1,



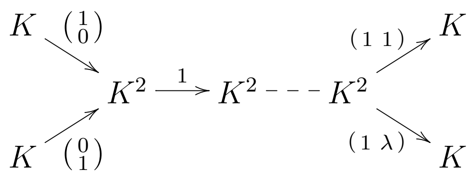
The type of the tubular family for $\mathcal{Q}$ is $(l, 2, 2)$ and the dimension vectors of the regular simples are

$$\alpha_r := e_r \quad \text{for } 1 \leq r < l, \quad \alpha_l := e_1 + \cdots + e_{l+3}$$

and

$$\beta_1 := e_1 + \cdots + e_{l-1} + e_l + e_{l+2}, \qquad \beta_2 := e_1 + \cdots + e_{l-1} + e_{l+1} + e_{l+3},$$
$$\gamma_1 := e_1 + \cdots + e_{l-1} + e_l + e_{l+3}, \qquad \gamma_2 := e_1 + \cdots + e_{l-1} + e_{l+1} + e_{l+2}.$$

The homogeneous regular simples $T(\lambda)$ are parametrised by $\lambda \in K - \{0, 1\}$. These are given by



The automorphism $\mathbf{a}$ fixes each α_r and swaps $\beta_1 \leftrightarrow \beta_2$ and $\gamma_1 \leftrightarrow \gamma_2$. Also, each $T(\lambda)$ is fixed (up to isomorphism). Therefore we have ii-indecomposables

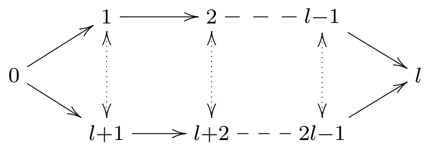
$$V(\alpha_r), \quad V(\beta_1) \oplus V(\beta_2), \quad V(\gamma_1) \oplus V(\gamma_2), \quad T(\lambda).$$

Hence

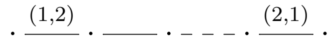
$$A\text{-}ii(m\delta, q) = q + l \quad \text{for all } m.$$

Case 3.

Let $(\mathcal{Q}, \mathbf{a})$ be given by the quiver $\widetilde{\mathbb{A}}_{2l-1}$ with $\mathbf{a}$ of order 2 as shown below.



Then Γ is the valued graph $\widetilde{\mathbb{B}}_l$, of tier number 2,



N.B. We do not need to consider the case when $\mathbf{a}^2(\rho) = \zeta^{2u}\rho$, since we can always reduce to the case when $\rho : 0 \to 1$. Now apply the inner automorphism given by conjugation by $1 + \zeta^{-u}\varepsilon_0$, noting that both arrows starting at 0 are in the same $\mathbf{a}$-orbit.

The tubular type of $\mathcal{Q}$ is (l, l) with dimension vectors

$$\begin{aligned} \alpha_r &:= e_r \quad \text{for } 1 \le r < l, \quad \alpha_l := e_0 + e_{l+1} + \cdots + e_{2l-1} + e_l, \\ \beta_r &:= e_{l+r} \quad \text{for } 1 \le r < l, \quad \beta_l := e_0 + e_1 + \cdots + e_{l-1} + e_l. \end{aligned}$$

The homogeneous regular simples $T(\lambda)$ are parametrised by $0 \neq \lambda \in K$. These have a copy of the field K at each vertex and each arrow acts as the identity except $0 \to 1$, which acts as multiplication by λ.

Now $\mathbf{a}$ acts by swapping $\alpha_r \leftrightarrow \beta_r$ and by $T(\lambda) \mapsto T(\lambda^{-1})$. Therefore we have ii-indecomposables

$$V(\alpha_r) \oplus V(\beta_r), \quad T(1), \quad T(-1), \quad T(\lambda) \oplus T(\lambda^{-1}) \quad \lambda \in K - \{0, \pm 1\}.$$

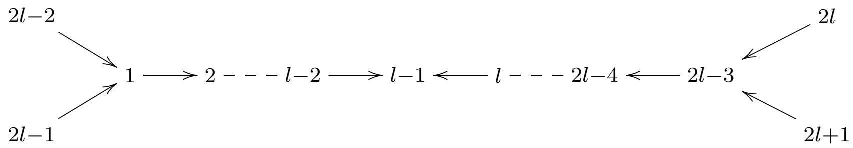
Now $T(\lambda) \oplus T(\lambda^{-1})$ is defined over $\mathbb{F}_q$ if and only if $\lambda + \lambda^{-1} \in \mathbb{F}_q$. Since $\lambda \neq 0, \pm 1$ this can take any value except ± 2 and so we get $q - 2$ different isomorphism classes of absolutely ii-indecomposables.

Hence

$$A\text{-}ii(m\delta, q) = \begin{cases} 2 & \text{if } m \not\equiv 0 \bmod 2; \\ q + l & \text{if } m \equiv 0 \bmod 2. \end{cases}$$

Case 4.

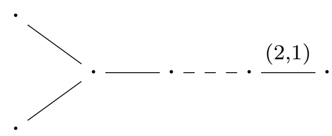
Let $\mathcal{Q}$ be the graph $\widetilde{\mathbb{D}}_{2l}$ for $l \geq 3$ with orientation



and let **a** be the automorphism of order 2 defined by reflection in the vertical line through vertex $l-1$. That is,

$$i \mapsto 2l-2-i \quad \text{for } 1 \leq i < 2l-2 \qquad \text{and} \qquad 2l-2 \leftrightarrow 2l, \quad 2l-1 \leftrightarrow 2l+1.$$

Then Γ is the valued graph $\widetilde{\mathbb{BD}}_l$, of tier number 1,



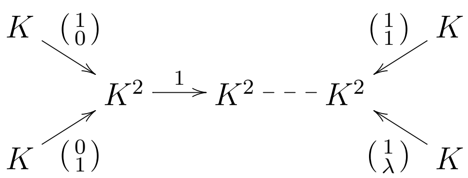
$\mathcal{Q}$ has tubular type $(2l-2, 2, 2)$ with dimension vectors

$$\begin{aligned}
&\alpha_r := e_r && 1 \leq r \leq l-2, && \alpha_{l-1} := e_l + \cdots + e_{2l-3} + e_{l-1} + e_{2l} + e_{2l+1},\\
&\alpha_r := e_{3l-3-r} && l \leq r \leq 2l-3, && \alpha_{2l-2} := e_1 + \cdots + e_{l-2} + e_{l-1} + e_{2l-2} + e_{2l-1}
\end{aligned}$$

and

$$\begin{aligned}
&\beta_1 := e_1 + \cdots + e_{2l-3} + e_{2l-2} + e_{2l}, && \beta_2 := e_1 + \cdots + e_{2l-3} + e_{2l-1} + e_{2l+1},\\
&\gamma_1 := e_1 + \cdots + e_{2l-3} + e_{2l-2} + e_{2l+1}, && \gamma_2 := e_1 + \cdots + e_{2l-3} + e_{2l-1} + e_{2l}.
\end{aligned}$$

The homogeneous regular simples $T(\lambda)$ are parametrised by $\lambda \in K - \{0,1\}$. These are given by



Now $\mathbf{a}$ acts as $\alpha_r \leftrightarrow \alpha_{l-1+r}$ for $1 \le r < l$ and $\gamma_1 \leftrightarrow \gamma_2$. Also, $V(\beta_r)$ and $T(\lambda)$ are fixed. Therefore we have ii-indecomposables

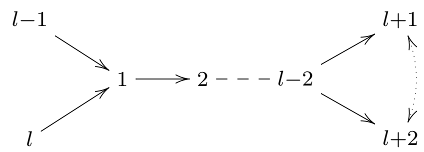
$$V(\alpha_r) \oplus V(\alpha_{l-1+r}) \quad \text{for } 1 \le r < l, \quad V(\gamma_1) \oplus V(\gamma_2), \quad V(\beta_r), \quad T(\lambda).$$

Hence

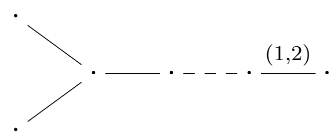
$$A\text{-}ii(m\delta, q) = q + l \quad \text{for all } m.$$

Case 5.

Let $(\mathcal{Q}, \mathbf{a})$ be given by the quiver $\widetilde{\mathbb{D}}_{l+1}$ with automorphism



Then Γ is given by the valued graph $\widetilde{\mathbb{C}\,\mathbb{D}}_l$, of tier number 2,



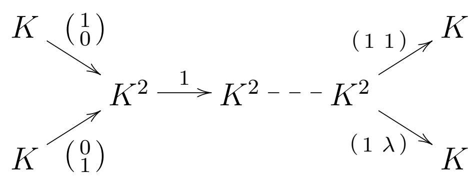
$\mathcal{Q}$ has tubular type $(l-1, 2, 2)$ with dimension vectors

$$\alpha_r := e_r \quad \text{for } 1 \le r < l-1, \quad \alpha_{l-1} := e_1 + \cdots + e_{l+2}$$

and

$$\beta_1 := e_1 + \cdots + e_{l-2} + e_{l-1} + e_{l+1}, \qquad \beta_2 := e_1 + \cdots + e_{l-2} + e_l + e_{l+2},$$
$$\gamma_1 := e_1 + \cdots + e_{l-2} + e_{l-1} + e_{l+2}, \qquad \gamma_2 := e_1 + \cdots + e_{l-2} + e_l + e_{l+1}.$$

The homogeneous regular simples $T(\lambda)$ are parametrised by $\lambda \in K - \{0, 1\}$. These are given by



We have that $\mathbf{a}$ fixes each α_r and swaps $\beta_r \leftrightarrow \gamma_r$. Also, $T(\lambda) \leftrightarrow T(\lambda^{-1})$. Therefore we get absolutely ii-indecomposables

$$V(\alpha_r), \quad V(\beta_r) \oplus V(\gamma_r), \quad T(-1), \quad T(\lambda) \oplus T(\lambda^{-1}) \quad \lambda \in K - \{0, \pm 1\}.$$

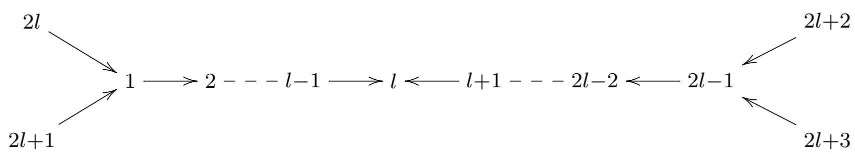
Again, $T(\lambda) \oplus T(\lambda^{-1})$ is defined over $\mathbb{F}_q$ if and only if $\lambda + \lambda^{-1} \in \mathbb{F}_q$, and since $\lambda \neq 0, \pm 1$, this can take any value except ± 2.

Hence

$$A\text{-}ii(m\delta, q) = \begin{cases} l & \text{if } m \not\equiv 0 \bmod 2; \\ q + l & \text{if } m \equiv 0 \bmod 2. \end{cases}$$

Case 6.

Let $\mathcal{Q}$ be the graph $\widetilde{\mathbb{D}}_{2(l+1)}$ for $l \geq 1$ with orientation



and let $\mathbf{a}$ be the automorphism of order 4 defined by

$$i \mapsto 2l - i \quad \text{for } 1 \leq i \leq l \qquad \text{and} \qquad 2l \mapsto 2l + 2 \mapsto 2l + 1 \mapsto 2l + 3.$$

Then Γ is the valued graph $\widetilde{\mathbb{B}\mathbb{C}}_l$, with tier number 2,

As in Case 4, this has type $(2l, 2, 2)$ with dimension vectors

$$\begin{aligned} \alpha_r &:= e_r && \text{for } 1 \leq r < l, & \alpha_l &:= e_{l+1} + \cdots + e_{2l-1} + e_l + e_{2l+2} + e_{2l+3}, \\ \alpha_r &:= e_{3l-r} && \text{for } l + 1 \leq r < 2l, & \alpha_{2l} &:= e_1 + \cdots + e_{l-1} + e_l + e_{2l} + e_{2l+1} \end{aligned}$$

and

$$\begin{aligned} \beta_1 &:= e_1 + \cdots + e_{2l-3} + e_{2l} + e_{2l+2}, & \beta_2 &:= e_1 + \cdots + e_{2l-3} + e_{2l+1} + e_{2l+3}, \\ \gamma_1 &:= e_1 + \cdots + e_{2l-3} + e_{2l} + e_{2l+3}, & \gamma_2 &:= e_1 + \cdots + e_{2l-3} + e_{2l+1} + e_{2l+2}. \end{aligned}$$

The homogeneous tubes are parametrised by $K - \{0, 1\}$ with regular simples $T(\lambda)$ analogous to those defined in Case 4.

The automorphism acts as

$$\alpha_r \mapsto \alpha_{r+l} \quad \text{for } 1 \leq r \leq l, \qquad \gamma_1 \mapsto \beta_1 \mapsto \gamma_2 \mapsto \beta_2, \qquad T(\lambda) \mapsto T(\lambda^{-1}).$$

Therefore we have ii-indecomposables

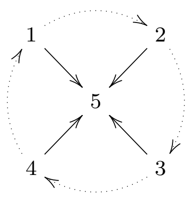
$$V(\alpha_r) \oplus V(\alpha_{r+l}), \quad V(\beta_1) \oplus V(\beta_2) \oplus V(\gamma_1) \oplus V(\gamma_2),$$
$$T(-1), \quad T(\lambda) \oplus T(\lambda^{-1}) \quad \text{for } \lambda \in K - \{0, \pm 1\}.$$

Hence

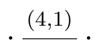
$$A\text{-}ii(m\delta, q) = \begin{cases} l + 1 & \text{if } m \not\equiv 0 \bmod 2; \\ q + l & \text{if } m \equiv 0 \bmod 2. \end{cases}$$

Case 7.

Let $(\mathcal{Q}, \mathbf{a})$ be the quiver $\widetilde{\mathbb{D}}_4$ with automorphism $\mathbf{a}$ of order 4



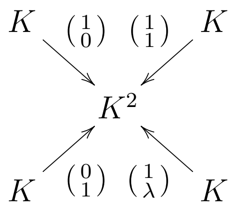
so that Γ is the valued graph $\widetilde{\mathbb{A}}_{11}$, of tier number 2,



$\mathcal{Q}$ has tubular type $(2, 2)$ with dimension vectors

$$\alpha_1 := e_1 + e_4 + e_5, \qquad \beta_1 := e_1 + e_2 + e_5, \qquad \gamma_1 := e_1 + e_3 + e_5,$$
$$\alpha_2 := e_2 + e_3 + e_5, \qquad \beta_2 := e_3 + e_4 + e_5, \qquad \gamma_2 := e_2 + e_4 + e_5.$$

The homogeneous regular simples $T(\lambda)$ are parametrised by $\lambda \in K - \{0, 1\}$ and are given by



The automorphism $\mathbf{a}$ acts as

$$\alpha_1 \mapsto \beta_1 \mapsto \alpha_2 \mapsto \beta_2, \qquad \gamma_1 \leftrightarrow \gamma_2 \qquad \text{and} \qquad T(\lambda) \mapsto T\left(\frac{\lambda}{\lambda - 1}\right).$$

We note that the automorphism $\lambda \mapsto \frac{\lambda}{\lambda-1}$ of $\mathbb{P}^1 K$ is of order 2. Therefore we have ii-indecomposables

$$V(\alpha_1) \oplus V(\alpha_2) \oplus V(\beta_1) \oplus V(\beta_2), \quad V(\gamma_1) \oplus V(\gamma_2),$$
$$T(2), \quad T(\lambda) \oplus T\left(\frac{\lambda}{\lambda - 1}\right) \quad \text{for } \lambda \in K - \{0, 1, 2\}.$$

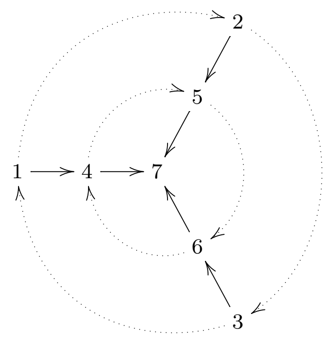
For $\lambda \in K - \{0, 1, 2\}$, the representation $T(\lambda) \oplus T(\frac{\lambda}{\lambda-1})$ is defined over $\mathbb{F}_q$ if and only if λ and $\frac{\lambda}{\lambda-1}$ are the roots of a polynomial defined over $\mathbb{F}_q$, which is if and only if $\frac{\lambda^2}{\lambda-1} \in \mathbb{F}_q$. This can take any value except 0 and 4 and so we get $q - 2$ such absolutely ii-indecomposables defined over $\mathbb{F}_q$.

Hence

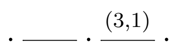
$$A\text{-}ii(m\delta, q) = \begin{cases} 2 & \text{if } m \not\equiv 0 \bmod 2; \\ q + 1 & \text{if } m \equiv 0 \bmod 2. \end{cases}$$

Case 8.

Let $(\mathcal{Q}, \mathbf{a})$ be the quiver $\widetilde{\mathbb{E}}_6$ with automorphism of order 3 indicated below



Then Γ is the valued graph $\widetilde{\mathbb{G}}_{22}$, of tier number 1,



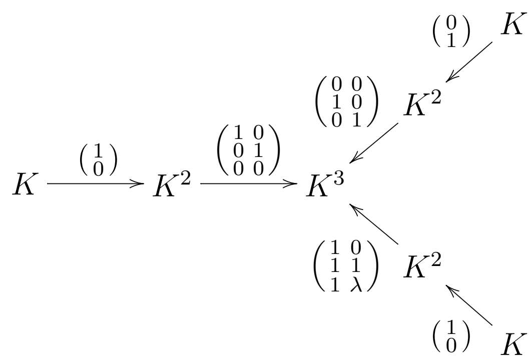
This has type $(2, 3, 3)$ with dimension vectors

$$\alpha_1 := e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + 2e_7, \quad \alpha_2 := e_4 + e_5 + e_6 + e_7$$

and

$$\beta_1 := e_1 + e_4 + e_7 + e_6, \qquad \gamma_1 := e_1 + e_4 + e_7 + e_5,$$
$$\beta_2 := e_2 + e_5 + e_7 + e_4, \qquad \gamma_2 := e_3 + e_6 + e_7 + e_4,$$
$$\beta_3 := e_3 + e_6 + e_7 + e_5, \qquad \gamma_3 := e_2 + e_5 + e_7 + e_6.$$

The homogeneous regular simples $T(\lambda)$ are parametrised by $\lambda \in K - \{0, 1\}$ and are given by



Now **a** fixes α_r, sends $\beta_r \mapsto \beta_{r+1}$ and $\gamma_r \mapsto \gamma_{r-1}$. Also, $T(\lambda)$ is fixed up to isomorphism. Therefore we have ii-indecomposables

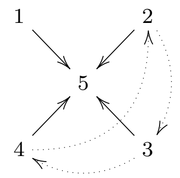
$$V(\alpha_r), \quad V(\beta_1) \oplus V(\beta_2) \oplus V(\beta_3), \quad V(\gamma_1) \oplus V(\gamma_2) \oplus V(\gamma_3), \quad T(\lambda).$$

Hence

$$A\text{-}ii(m\delta, q) = q + 2 \quad \text{for all } m.$$

Case 9.

Let $(\mathcal{Q}, \mathbf{a})$ be the quiver $\widetilde{\mathbb{D}}_4$ with $\mathbf{a}$ of order 3 given by



Then Γ is the valued graph $\widetilde{\mathbb{G}}_{21}$, of tier number 3,

$$\cdot \;\text{———}\; \cdot \;\overset{(1,3)}{\text{———}}\; \cdot$$

Using the same notation for the regular simples as in Case 7, we have that the automorphism **a** acts as

$$\alpha_r \mapsto \beta_r \mapsto \gamma_r, \qquad T(\lambda) \mapsto T\left(\frac{1}{1-\lambda}\right) \mapsto T\left(\frac{\lambda-1}{\lambda}\right).$$

Therefore we have the ii-indecomposables

$$V(\alpha_r) \oplus V(\beta_r) \oplus V(\gamma_r), \quad T(-\zeta), \quad T(-\zeta^2),$$
$$T(\lambda) \oplus T\left(\frac{1}{1-\lambda}\right) \oplus T\left(\frac{\lambda-1}{\lambda}\right) \quad \text{for } \lambda \in K - \{0, 1, -\zeta, -\zeta^2\},$$

where ζ is a primitive cube root of unity.

We are assuming that $q \equiv 1 \bmod 3$, so that $\zeta \in \mathbb{F}_q$. Also, $T(\lambda) \oplus T(\frac{1}{1-\lambda}) \oplus T(\frac{\lambda-1}{\lambda})$ is defined over $\mathbb{F}_q$ if and only either $\lambda \in \mathbb{F}_q$ or $\lambda^q = \frac{1}{1-\lambda}$ or $\lambda^q = \frac{\lambda-1}{\lambda}$. Since

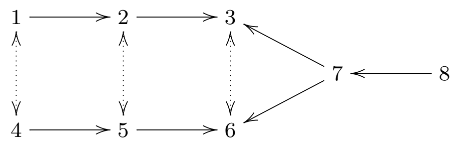
$x^{q+1} - x^q + 1$ is separable over $\mathbb{F}_q$, there are $q+1$ solutions in K. Similarly for $x^{q+1} - x + 1$, and the only common solutions are $-\zeta$ and $-\zeta^2$, which are the only two solutions in $\mathbb{F}_q$. Since we are assuming $\lambda \neq 0, 1, -\zeta, -\zeta^2$, we obtain $\frac{1}{3}(q - 4 + q - 1 + q - 1) = q - 2$ absolutely ii-indecomposables defined over $\mathbb{F}_q$.

Hence

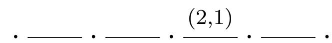
$$A(m\delta, q) = \begin{cases} 2 & \text{if } m \not\equiv 0 \bmod 3; \\ q+2 & \text{if } m \equiv 0 \bmod 3. \end{cases}$$

Case 10.

Let $(\mathcal{Q}, \mathbf{a})$ be the quiver $\widetilde{\mathbb{E}}_7$ with $\mathbf{a}$ of order 2



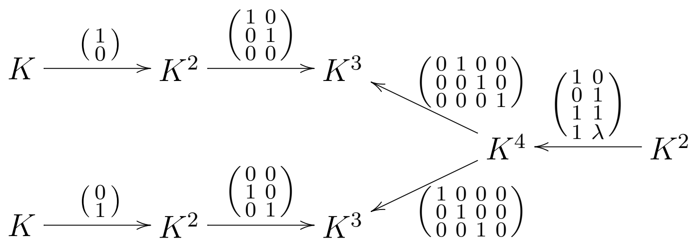
Then Γ is the valued graph $\widetilde{\mathbb{F}}_{42}$, of tier number 1,



This has type $(2, 3, 4)$ with dimension vectors

$$\alpha_1 := e_1 + e_2 + 2e_3 + e_5 + e_6 + 2e_7 + e_8,$$
$$\alpha_2 := e_2 + e_3 + e_4 + e_5 + 2e_6 + 2e_7 + e_8,$$
$$\beta_1 := e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7,$$
$$\beta_2 := e_2 + e_3 + e_5 + e_6 + e_7 + e_8,$$
$$\beta_3 := e_3 + e_6 + 2e_7 + e_8,$$
$$\gamma_1 := e_1 + e_2 + e_3 + e_6 + e_7 + e_8,$$
$$\gamma_2 := e_2 + e_3 + e_7,$$
$$\gamma_3 := e_3 + e_4 + e_5 + e_6 + e_7 + e_8,$$
$$\gamma_4 := e_5 + e_6 + e_7.$$

The homogeneous regular simples $T(\lambda)$ are parametrised by $\lambda \in K - \{0, 1\}$ and are given by



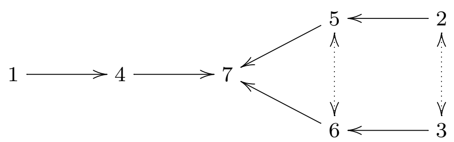
Now **a** fixes β_r and swaps $\alpha_1 \leftrightarrow \alpha_2$ and $\gamma_r \leftrightarrow \gamma_{r+2}$ for $r = 1, 2$. Also, each $T(\lambda)$ is fixed up to isomorphism.

Hence

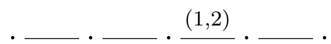
$$A\text{-}ii(m\delta, q) = q + 4 \quad \text{for all } m.$$

Case 11.

Let $(\mathcal{Q}, \mathbf{a})$ be the quiver $\widetilde{\mathbb{E}}_6$ with **a** of order 2



Then Γ is the valued graph $\widetilde{\mathbb{F}}_{41}$, of tier number 2,



We use the same notation as in Case 9. Then **a** acts by fixing α_r and swapping $\beta_r \leftrightarrow \gamma_r$. Also ${}^{\mathbf{a}}(T(\lambda))$ is isomorphic to $T(\frac{\lambda}{\lambda-1})$. Therefore we have ii-indecomposables

$$V(\alpha_r), \quad V(\beta_r) \oplus V(\gamma_r), \quad T(2), \quad T(\lambda) \oplus T\left(\frac{\lambda}{\lambda - 1}\right) \quad \text{for } \lambda \in K - \{0, 1, 2\}.$$

Hence

$$A\text{-}ii(m\delta, q) = \begin{cases} 3 & \text{if } m \not\equiv 0 \bmod 2; \\ q + 4 & \text{if } m \equiv 0 \bmod 2. \end{cases}$$

6.3 Summary

We can summarise the preceding results as follows.

Let $(\mathcal{Q}, \mathbf{a})$ be an affine quiver with automorphism of order n; let Γ be the associated valued graph and let $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ be the dual quiver with automorphism. Define r by $h(\delta_{\widetilde{\mathcal{Q}}}) = r\delta_\Gamma$.

We note that in each case except Case 1, r equals the tier number of Γ, whereas in Case 1, $r = n/t$ but the tier number is 1.

In [25], Kac calculates the multiplicities of the roots $m\delta$ in the Kac-Moody Lie algebra $\mathfrak{g}(\Gamma)$. Comparing these numbers to our polynomials we obtain

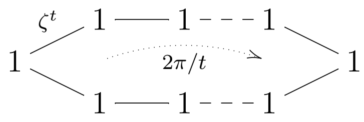
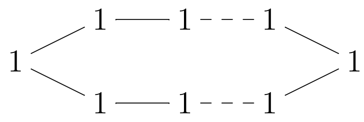
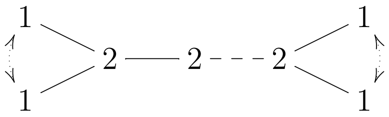
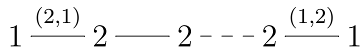
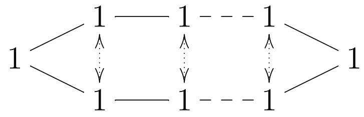
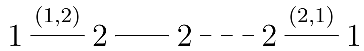
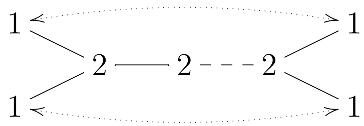
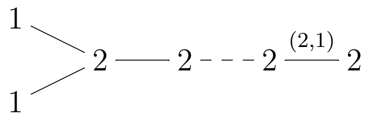
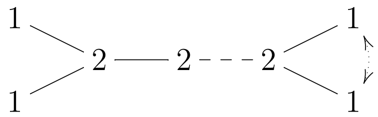
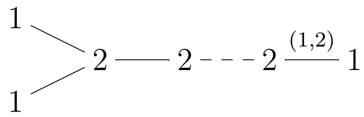
Theorem 6.3.1.

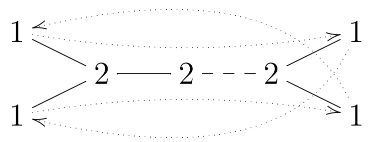
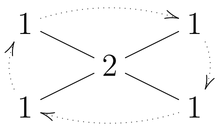
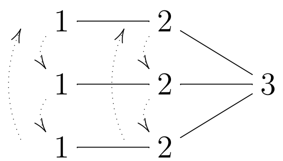
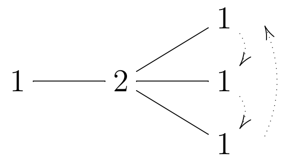
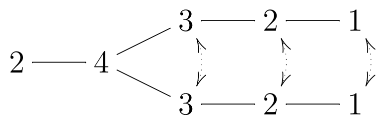
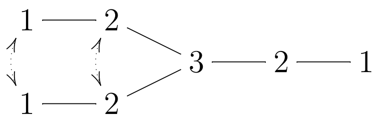
$$A\text{-}ii(m\delta, q) = \begin{cases} 1 + \dim \mathfrak{g}(\Gamma)_{m\delta} & \textit{if } m \not\equiv 0 \bmod r; \\ q + \dim \mathfrak{g}(\Gamma)_{m\delta} & \textit{if } m \equiv 0 \bmod r. \end{cases}$$

N.B. In every case, we observe that the coefficients of the polynomials $A\text{-}ii(\alpha, q)$ are all non-negative integers.

We now combine all this information into a table. We use the following notation: l is the rank of the GCM for Γ (so Γ has $l + 1$ vertices); the numbers at the vertices indicate the dimension vector δ; the tier number always refers to the tier number of Γ.

Also, we retain the same order of the cases as in the main text.

$(\mathcal{Q}, \mathbf{a})$	Γ	tier no.	$A\text{-}ii(m\delta, q)$
		1	$\begin{cases} l+1 & m \not\equiv 0 \bmod n/t \\ q+l & m \equiv 0 \bmod n/t \end{cases}$
		1	$q+l$
		2	$\begin{cases} 2 & m \not\equiv 0 \bmod 2 \\ q+l & m \equiv 0 \bmod 2 \end{cases}$
		1	$q+l$
		2	$\begin{cases} l & m \not\equiv 0 \bmod 2 \\ q+l & m \equiv 0 \bmod 2 \end{cases}$

	$1 \overset{(2,1)}{\text{——}} 2 \text{——} 2 \text{- - -} 2 \overset{(2,1)}{\text{——}} 2$	2	$\begin{cases} l+1 & m \not\equiv 0 \bmod 2 \\ q+l & m \equiv 0 \bmod 2 \end{cases}$
	$1 \overset{(4,1)}{\text{——}} 2$	2	$\begin{cases} 2 & m \not\equiv 0 \bmod 2 \\ q+1 & m \equiv 0 \bmod 2 \end{cases}$
	$1 \text{——} 2 \overset{(3,1)}{\text{——}} 3$	1	$q+2$
	$1 \text{——} 2 \overset{(1,3)}{\text{——}} 1$	3	$\begin{cases} 2 & m \not\equiv 0 \bmod 3 \\ q+2 & m \equiv 0 \bmod 3 \end{cases}$
	$2 \text{——} 4 \overset{(1,2)}{\text{——}} 3 \text{——} 2 \text{——} 1$	1	$q+4$
	$1 \text{——} 2 \overset{(2,1)}{\text{——}} 3 \text{——} 2 \text{——} 1$	2	$\begin{cases} 3 & m \not\equiv 0 \bmod 2 \\ q+4 & m \equiv 0 \bmod 2 \end{cases}$

Chapter 7

Kac-Moody Lie Algebras

Given any symmetrisable GCM $C = D^{-1}B$, we can define a Kac-Moody Lie algebra $\mathfrak{g}(C)$, unique up to isomorphism. (If C corresponds to the valued graph Γ, then we will also write $\mathfrak{g}(\Gamma)$.) Furthermore, if we have a quiver with automorphism $(\mathcal{Q}, \mathbf{a})$, then $\mathbf{a}$ acts naturally on the Chevalley generators of $\mathfrak{g}(\mathcal{Q})$ and so $\mathbf{a}$ determines an algebra automorphism of the derived subalgebra $\mathfrak{g}(\mathcal{Q})'$.

This action can be extended to an automorphism σ of the whole Lie algebra and the resulting fixed-point subalgebra $\mathcal{L}$ is a so-called generalised Kac-Moody Lie algebra. However, for certain choices of σ there is a natural embedding of the symmetrisable Kac-Moody Lie algebra $\mathfrak{g}(C^{\mathrm{tr}}) \hookrightarrow \mathcal{L}$, giving $\mathcal{L}$ the structure of an integrable $\mathfrak{g}(C^{\mathrm{tr}})$-module. It is this structure that we shall be most interested in. We prove that the non-zero weights are precisely the roots of $\mathfrak{g}(C^{\mathrm{tr}})$ and that the real roots all occur with multiplicity one.

Finally, we explicitly calculate the structure of this module when $\mathcal{Q}$ is affine and observe that the weight multiplicities for $\mathcal{L}$ coincide with the constant terms of the polynomials $A\text{-}ii(\alpha, q)$ determined in the previous chapter. This suggests the following conjecture, generalising a conjecture of Kac (c.f. Section 1.6).

Let $(\mathcal{Q}, \mathbf{a})$ be a quiver with automorphism, Γ the associated valued graph and $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ the dual quiver with automorphism. Let σ be any algebra automorphism of $\mathfrak{g}(\widetilde{\mathcal{Q}})$ lifting the action of $\tilde{\mathbf{a}}$ on the derived subalgebra and such that $\mathfrak{g}(\Gamma)$ embeds in the fixed-point algebra $\mathcal{L} := \mathfrak{g}(\widetilde{\mathcal{Q}})^{\langle\sigma\rangle}$. We view $\mathcal{L}$ as an integrable $\mathfrak{g}(\Gamma)$-module.

Conjecture 2′. $A\text{-}ii(\alpha, 0) = \dim \mathcal{L}_\alpha$.

7.1 Automorphisms of Kac-Moody Lie Algebras

Let $(\mathcal{Q}, \mathbf{a})$ be a quiver with admissible automorphism, Γ the associated valued graph and $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ the dual quiver with automorphism. We write A, $C = D^{-1}B$ and $\widetilde{A}$ for the GCMs of $\mathcal{Q}$, Γ and $\widetilde{\mathcal{Q}}$ respectively. In this section we show that the automorphism $\tilde{\mathbf{a}}$ acts canonically on the derived Lie algebra $\mathfrak{g}(\widetilde{\mathcal{Q}})'$ and that there is a natural embedding of the Lie algebra $\mathfrak{g}(\Gamma)'$ into the fixed point subalgebra $\big(\mathfrak{g}(\widetilde{\mathcal{Q}})'\big)^{\langle\tilde{\mathbf{a}}\rangle}$.

In fact, we describe how we can lift $\tilde{\mathbf{a}}$ to an automorphism σ of the whole Kac-Moody Lie algebra $\mathfrak{g}(\widetilde{\mathcal{Q}})$ so that $\mathfrak{g}(\Gamma)$ embeds in the fixed-point subalgebra $\mathcal{L} := \mathfrak{g}(\widetilde{\mathcal{Q}})^{\langle\sigma\rangle}$.

Lemma 7.1.1. *The automorphism $\tilde{\mathbf{a}}$ acts naturally on $\mathfrak{h}(\widetilde{\mathcal{Q}})'$ by $H_i \mapsto H_{\tilde{\mathbf{a}}(i)}$ and the canonical map $\mathfrak{h}(\Gamma)' \to \big(\mathfrak{h}(\widetilde{\mathcal{Q}})'\big)^{\langle\tilde{\mathbf{a}}\rangle}$ sending $H_{\mathbf{i}} \to \sum_\mu H_{(\mathbf{i},\mu)}$ is an isomorphism. Moreover, the restrictions of the bilinear forms satisfy $(-,-)_\Gamma = \frac{1}{n}(-,-)_{\widetilde{\mathcal{Q}}}$.*

Proof.
The first statement is clear. Also, by equation (3.4.5), we have

$$\sum_{\mu,\nu}(H_{(\mathbf{i},\mu)}, H_{(\mathbf{j},\nu)})_{\widetilde{\mathcal{Q}}} = \sum_{\mu,\nu} \tilde{a}_{(\mathbf{i},\mu)(\mathbf{j},\nu)} = \frac{n}{d_{\mathbf{i}}d_{\mathbf{j}}} b_{\mathbf{ij}},$$

which equals $n(H_{\mathbf{i}}, H_{\mathbf{j}})_\Gamma$.
∎

In particular, there is an isomorphism $\mathfrak{c}(\Gamma) \xrightarrow{\sim} \mathfrak{c}(\widetilde{\mathcal{Q}})^{\langle\tilde{\mathbf{a}}\rangle}$, so both spaces have dimension $\operatorname{corank} C$.

We now wish to extend the automorphism $\tilde{\mathbf{a}}$ of $\mathfrak{g}(\widetilde{\mathcal{Q}})'$ to an automorphism σ of $\mathfrak{g}(\widetilde{\mathcal{Q}})$ such that there is a monomorphism $\mathfrak{g}(\Gamma) \hookrightarrow \mathfrak{g}(\widetilde{\mathcal{Q}})^{\langle\sigma\rangle}$. We also describe which choices of σ we can make.

Let $\mathrm{DAut}(\mathfrak{g})$ be the subgroup of $\mathrm{Aut}(\mathfrak{g})$ consisting of the diagram automorphisms — that is, the automorphisms preserving each of the sets $\mathfrak{h}$, $\{E_i\}$ and $\{F_i\}$.

Proposition 7.1.2. *There is a short exact sequence*

$$0 \to \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}/\mathfrak{h}', \mathfrak{c}) \to \mathrm{DAut}(\mathfrak{g}) \to \mathrm{Aut}(A) \to 0,$$

where $\mathfrak{g}$ *is a Kac-Moody Lie algebra with symmetric GCM* A *and* $\mathrm{Aut}(A)$ *is the set of permutations* $\mathbf{a}$ *such that* $a_{\mathbf{a}(i)\mathbf{a}(j)} = a_{ij}$.

The following proof is taken from [26].

Proof.
If $\sigma \in \mathrm{DAut}(\mathfrak{g})$, then $\sigma(H_i) = \sigma[E_i, F_i] = [\sigma(E_i), \sigma(F_i)]$ and so there exists a permutation $\mathbf{a}$ of $\mathcal{I}$ such that $\sigma(E_i) = E_{\mathbf{a}(i)}$, $\sigma(F_i) = F_{\mathbf{a}(i)}$ and $\sigma(H_i) = H_{\mathbf{a}(i)}$. In fact, by applying σ to $[H_i, E_j] = a_{ij}E_j$, we see that $\mathbf{a} \in \mathrm{Aut}(A)$. Moreover, each $\mathbf{a} \in \mathrm{Aut}(A)$ is so obtained. For, let $\Lambda = \mathbb{C}\mathcal{I}$ be the subspace of $\mathfrak{h}^*$ spanned by the e_i. Then the quotient space $\mathfrak{h}/\mathfrak{c}$ is dual to Λ and so the natural action of $\mathrm{Aut}(A)$ on Λ, namely $\mathbf{a} : e_i \mapsto e_{\mathbf{a}(i)}$, defines an action of $\mathrm{Aut}(A)$ on $\mathfrak{h}/\mathfrak{c}$. This must map H_i mod $\mathfrak{c}$ to $H_{\mathbf{a}(i)}$ mod $\mathfrak{c}$ and so the subspace $\mathfrak{h}'/\mathfrak{c}$ is $\mathrm{Aut}(A)$-stable.

Now, as $\mathrm{Aut}(A)$ is finite, there exists $\mathfrak{h}''$ such that $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ and $(\mathfrak{h}'' + \mathfrak{c})/\mathfrak{c}$ is $\mathrm{Aut}(A)$-stable. So, for any $\mathbf{a} \in \mathrm{Aut}(A)$, we can define an automorphism σ of $\mathfrak{h}$ by $\sigma(H_i) := H_{\mathbf{a}(i)}$ and $\sigma|_{\mathfrak{h}''}$ is the pull-back of $\mathbf{a}$ on $(\mathfrak{h}'' + \mathfrak{c})/\mathfrak{c}$. By construction, $e_{\mathbf{a}(i)}(\sigma(H)) = e_i(H)$ for all $H \in \mathfrak{h}$ and so we can extend σ to an automorphism of $\mathfrak{g}$ via $E_i \mapsto E_{\mathbf{a}(i)}$ and $F_i \mapsto F_{\mathbf{a}(i)}$.

The kernel of the map $\mathrm{DAut}(\mathfrak{g}) \to \mathrm{Aut}(A)$ is $\mathrm{Aut}(\mathfrak{g}; \mathfrak{g}')$, the subgroup consisting of all automorphisms acting trivially on $\mathfrak{g}'$. If $\sigma \in \mathrm{Aut}(\mathfrak{g}; \mathfrak{g}')$, then $\mathfrak{h}$ must be σ-stable. Also,

$$e_i(H)E_i = \sigma[H, E_i] = [\sigma(H), E_i] = e_i(\sigma(H))E_i.$$

Therefore there exists $\varphi : \mathfrak{h}'' \to \mathfrak{c}$ such that $\sigma(H) = H + \varphi(H)$.

Conversely, given $\varphi : \mathfrak{h}'' \to \mathfrak{c}$, we can define $\sigma \in \mathrm{Aut}(\mathfrak{g}; \mathfrak{g}')$ by $\sigma(H) = H + \varphi(H)$ for all $H \in \mathfrak{h}''$. Thus $\mathrm{Aut}(\mathfrak{g}; \mathfrak{g}') \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}'', \mathfrak{c}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}/\mathfrak{h}', \mathfrak{c})$.
■

For ease of notation, henceforth in this section all Lie algebras will refer to $\widetilde{\mathcal{Q}}$ unless otherwise stated.

Consider the following subspace of $\mathfrak{h}$.

$$M := \{H \mid e_i(H) = e_{\tilde{\mathbf{a}}(i)}(H) \text{ for all } i \in \widetilde{\mathcal{I}}\} = \mathrm{ann}\big(\mathrm{span}\{e_i - e_{\tilde{\mathbf{a}}(i)} \mid i \in \widetilde{\mathcal{I}}\}\big). \tag{7.1.1}$$

Then M contains the centre $\mathfrak{c}$ and $M/\mathfrak{c} = \big(\mathfrak{h}/\mathfrak{c}\big)^{\langle \tilde{\mathbf{a}} \rangle}$. Thus for any lift σ of $\tilde{\mathbf{a}}$, $M^{\langle \sigma \rangle} = \mathfrak{h}^{\langle \sigma \rangle}$.

Let us write r and s for the coranks of $\widetilde{A}$ and C respectively, so $\dim \mathfrak{h} = |\widetilde{\mathcal{I}}| + r$.

Lemma 7.1.3. *M has dimension $|\mathbf{I}| + r$ and $M \cap \mathfrak{h}'$ has dimension $|\mathbf{I}| + r - s$.*

Proof.
Let $S = \mathrm{span}\{e_i - e_{\tilde{\mathbf{a}}(i)} \mid i \in \widetilde{\mathcal{I}}\}$. This has basis $e_{(\mathbf{i},\mu)} - e_{(\mathbf{i},\mu+1)}$ with $\mathbf{i} \in \mathbf{I}$ and $0 \leq \mu < (n/d_{\mathbf{i}}) - 1$ and so has dimension $|\widetilde{\mathcal{I}}| - |\mathbf{I}|$. Therefore $\dim M = \dim \mathfrak{h} - \dim S = |\mathbf{I}| + r$.

We lift the $\tilde{\mathbf{a}}$-action on $\mathfrak{h}'/\mathfrak{c}$ to $\mathfrak{h}'$ by mapping H_i to $H_{\tilde{\mathbf{a}}(i)}$. Then $(M \cap \mathfrak{h}')/\mathfrak{c} = (\mathfrak{h}'/\mathfrak{c})^{\langle \tilde{\mathbf{a}} \rangle}$ is isomorphic to $(\mathfrak{h}')^{\langle \tilde{\mathbf{a}} \rangle}/\mathfrak{c}^{\langle \tilde{\mathbf{a}} \rangle}$. Now $\dim(\mathfrak{h}')^{\langle \tilde{\mathbf{a}} \rangle} = |\mathbf{I}|$ and $\dim \mathfrak{c}^{\langle \tilde{\mathbf{a}} \rangle} = s$ by Lemma 7.1.1 and the subsequent remark. Thus $\dim M \cap \mathfrak{h}' = |\mathbf{I}| - s + r$.
■

Since $(\mathfrak{h}'' + \mathfrak{c})/\mathfrak{c}$ is also $\tilde{\mathbf{a}}$-stable, we deduce that $\big((\mathfrak{h}'' + \mathfrak{c})/\mathfrak{c}\big)^{\langle \tilde{\mathbf{a}} \rangle}$ has dimension s. Therefore, we can find linearly independent elements $X_1, \ldots, X_s \in M \cap \mathfrak{h}''$ with each X_i mod $\mathfrak{c}$ fixed by $\tilde{\mathbf{a}}$. These actually form a basis for $M \cap \mathfrak{h}''$.

Summing up, we have the following spaces

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}'' \qquad \text{and} \qquad M = (M \cap \mathfrak{h}') \oplus (M \cap \mathfrak{h}'') \tag{7.1.2}$$

of dimensions

$$\begin{aligned} \dim \mathfrak{h} &= |\widetilde{\mathcal{I}}| + r, & \dim \mathfrak{h}' &= |\widetilde{\mathcal{I}}|, & \dim \mathfrak{h}'' &= r, \\ \dim M &= |\mathbf{I}| + r, & \dim M \cap \mathfrak{h}' &= |\mathbf{I}| + r - s, & \dim M \cap \mathfrak{h}'' &= s. \end{aligned} \tag{7.1.3}$$

Now let

$$L := \operatorname{span}\{H_i - H_{\tilde{\mathbf{a}}(i)} \mid i \in \widetilde{\mathcal{I}}\}. \tag{7.1.4}$$

Proposition 7.1.4. *Let σ be any lift of $\tilde{\mathbf{a}}$ to $\mathfrak{g}$, corresponding to $\varphi : \mathfrak{h}'' \to \mathfrak{c}$. Then $(M^{\langle\sigma\rangle}, \{H_{\mathbf{i}}\}, \{e_{\mathbf{i}}\})$ is a realisation of C if and only if $\varphi(M \cap \mathfrak{h}'') \subset L \cap \mathfrak{c}$, where $H_{\mathbf{i}} := \sum_\mu H_{(\mathbf{i},\mu)}$ and $e_{\mathbf{i}} := \frac{d_{\mathbf{i}}}{n} \sum_\mu e_{(\mathbf{i},\mu)}$.*

Proof.
We know that the $H_{\mathbf{i}} := \sum_\mu H_{(\mathbf{i},\mu)}$ for a basis for $(M \cap \mathfrak{h}')^{\langle\sigma\rangle}$. Therefore the space of fixed-points has dimension $|\mathbf{I}| + s$ if and only if we can find $Y_1 \ldots, Y_s \in M^{\langle\sigma\rangle}$ spanning a complementary subspace to $M \cap \mathfrak{h}'$.

If we have such elements Y_i, then we can write $X_i = \sum_{j=1}^s p_{ij} Y_j - \sum_{j \in \widetilde{\mathcal{I}}} q_{ij} H_j$. Thus

$$\varphi(X_i) = \sigma(X_i) - X_i = \sum_j q_{ij}(H_j - H_{\tilde{\mathbf{a}}(j)}) \in L.$$

Conversely, if $\varphi(X_i) = \sum_j q_{ij}(H_j - H_{\tilde{\mathbf{a}}(j)}) \in L \cap \mathfrak{c}$, then $Y_i = X_i + \sum_j q_{ij} H_j$ is fixed by σ. Also, $\sum_j q_{ij} H_j \in M$ since

$$\begin{aligned} 0 = e_{\tilde{\mathbf{a}}(k)}\Big(\sum_j q_{ij}(H_j - H_{\tilde{\mathbf{a}}(j)})\Big) &= \sum_j q_{ij} e_{\tilde{\mathbf{a}}(k)}(H_j) - \sum_j q_{ij} e_{\tilde{\mathbf{a}}(k)}(H_{\tilde{\mathbf{a}}(j)}) \\ &= \sum_j q_{ij} e_{\tilde{\mathbf{a}}(k)}(H_j) - \sum_j q_{ij} e_k(H_j) = (e_{\tilde{\mathbf{a}}(k)} - e_k)\Big(\sum_j q_{ij} H_j\Big). \end{aligned}$$

Therefore $Y_i \in M$.

Now, by (3.4.5),

$$e_{\mathbf{j}}(H_{\mathbf{i}}) = \frac{d_{\mathbf{j}}}{n} \sum_{\mu,\nu} e_{(\mathbf{j},\nu)}(H_{(\mathbf{i},\mu)}) = \frac{d_{\mathbf{j}}}{n} \sum_{\mu,\nu} \tilde{a}_{(\mathbf{i},\mu)(\mathbf{j},\nu)} = \frac{1}{d_{\mathbf{i}}} b_{\mathbf{ij}} = c_{\mathbf{ij}}.$$

Therefore it only remains to show that the $e_{\mathbf{i}}$ are linearly independent modulo $\mathrm{ann}(M^{\langle\sigma\rangle})$. Let $\alpha = \sum_{\mathbf{j}} \lambda_{\mathbf{j}} e_{\mathbf{j}} \in \mathrm{ann}(M^{\langle\sigma\rangle})$. Then

$$0 = \alpha(H_{\mathbf{i}}) = \sum_{\mathbf{j}} \lambda_{\mathbf{j}} e_{\mathbf{j}}(H_{\mathbf{i}}) = \sum_{\mathbf{j}} c_{\mathbf{ij}} \lambda_{\mathbf{j}}.$$

Moreover, $e_{\mathbf{j}}(H_{(\mathbf{i},\mu)}) = \frac{d_{\mathbf{j}}}{n} \sum_{\nu} e_{(\mathbf{j},\nu)}(H_{(\mathbf{i},\mu)}) = \frac{1}{n} b_{\mathbf{ij}}$, so that

$$\alpha(H_{(\mathbf{i},\mu)}) = \sum_{\mathbf{j}} \lambda_{\mathbf{j}} e_{\mathbf{j}}(H_{(\mathbf{i},\mu)}) = \frac{1}{n} \sum_{\mathbf{j}} b_{\mathbf{ij}} \lambda_{\mathbf{j}} = \frac{d_{\mathbf{i}}}{n} \sum_{\mathbf{j}} c_{\mathbf{ij}} \lambda_{\mathbf{j}} = 0.$$

Hence

$$\alpha \in \mathrm{ann}(M^{\langle\sigma\rangle} + \mathfrak{h}') = \mathrm{ann}(M + \mathfrak{h}') \subset \mathrm{ann}(M) = \mathrm{span}\{e_i - e_{\tilde{\mathbf{a}}(i)} \mid i \in \widetilde{\mathcal{I}}\}.$$

However, $\tilde{\mathbf{a}}(\alpha) = \alpha$ but $\tilde{\mathbf{a}}$ does not have 1 as an eigenvalue on $\mathrm{ann}(M)$. Thus $\alpha = 0$.

■

N.B. We can rewrite this condition as $\varphi\big((M + \mathfrak{h}')/\mathfrak{h}'\big) \subset L \cap \mathfrak{c}$, where now $\varphi : \mathfrak{h}/\mathfrak{h}' \to \mathfrak{c}$.

Theorem 7.1.5. *There is a canonical monomorphism*

$$\mathfrak{g}(\Gamma)' \hookrightarrow (\mathfrak{g}')^{\langle\tilde{\mathbf{a}}\rangle}.$$

Moreover, if σ is an automorphism of $\mathfrak{g}$ extending $\tilde{\mathbf{a}}$, corresponding to $\varphi : \mathfrak{h}/\mathfrak{h}' \to \mathfrak{c}$ with $\varphi\big((M + \mathfrak{h}')/\mathfrak{h}'\big) \subset L \cap \mathfrak{c}$, then we can lift this monomorphism to give

$$\mathfrak{g}(\Gamma) \hookrightarrow \mathfrak{g}^{\langle\sigma\rangle}.$$

Proof.

Consider the elements $E_{\mathbf{i}} := \sum_{\mu} E_{(\mathbf{i},\mu)}$ (and similarly $F_{\mathbf{i}}$ and $H_{\mathbf{i}}$) in $(\mathfrak{g}')^{\langle\tilde{\mathbf{a}}\rangle}$. We show that these satisfy the Serre relations.

Clearly $[H_{\mathbf{i}}, H_{\mathbf{j}}] = 0$ and

$$[E_{\mathbf{i}}, F_{\mathbf{j}}] = \sum_{\mu,\nu} [E_{(\mathbf{i},\mu)}, F_{(\mathbf{j},\nu)}] = \delta_{\mathbf{ij}} \sum_{\mu} H_{(\mathbf{i},\mu)} = \delta_{\mathbf{ij}} H_{\mathbf{i}}.$$

Also,

$$[H_{\mathbf{i}}, E_{\mathbf{j}}] = \sum_{\mu,\nu}[H_{(\mathbf{i},\mu)}, E_{(\mathbf{j},\nu)}] = \sum_{\mu,\nu} \tilde{a}_{(\mathbf{i},\mu)(\mathbf{j},\nu)} E_{(\mathbf{j},\nu)} = c_{\mathbf{ij}} \sum_{\nu} E_{(\mathbf{j},\nu)} = c_{\mathbf{ij}} E_{\mathbf{j}}$$

and similarly $[H_{\mathbf{i}}, F_{\mathbf{j}}] = -c_{\mathbf{ij}} F_{\mathbf{j}}$.

Note that $\operatorname{ad} E_{(\mathbf{i},\mu)}$ and $\operatorname{ad} E_{(\mathbf{i},\nu)}$ commute for all μ, ν. Thus

$$(\operatorname{ad} E_{\mathbf{i}})^m = \sum_{|\lambda|=m} \binom{m}{\lambda} \prod_{\mu} (\operatorname{ad} E_{(\mathbf{i},\mu)})^{\lambda_\mu},$$

where we are summing over partitions $\lambda = (\lambda_\mu)$ and using multinomial coefficients. Consider $\prod_\mu (\operatorname{ad} E_{(\mathbf{i},\mu)})^{\lambda_\mu} E_{(\mathbf{j},\nu)}$. If $|\lambda| = 1 - c_{\mathbf{ij}}$, then $\lambda_\mu \geq 1 - \tilde{a}_{(\mathbf{i},\mu)(\mathbf{j},\nu)}$ for some μ and so $(\operatorname{ad} E_{(\mathbf{i},\mu)})^{\lambda_\mu} E_{(\mathbf{j},\nu)} = 0$. It follows that $(\operatorname{ad} E_{\mathbf{i}})^{1-c_{\mathbf{ij}}} E_{\mathbf{j}} = 0$, and similarly $(\operatorname{ad} F_{\mathbf{i}})^{1-c_{\mathbf{ij}}} F_{\mathbf{j}} = 0$.

Therefore there is a non-zero homomorphism $\mathfrak{g}(\Gamma)' \to (\mathfrak{g}')^{\langle \tilde{\mathbf{a}} \rangle}$. In particular, if C is non-degenerate, then $\mathfrak{g}(\Gamma)$ equals its derived subalgebra and is a direct sum of simple Lie algebras. Hence we have a monomorphism

$$\mathfrak{g}(\Gamma) \hookrightarrow (\mathfrak{g}')^{\langle \tilde{\mathbf{a}} \rangle} \subset \mathfrak{g}^{\langle \sigma \rangle}.$$

In the general case, we know that $M^{\langle \sigma \rangle}$ determines a realisation of C, so there is an isomorphism $\mathfrak{h}(\Gamma) \xrightarrow{\sim} M^{\langle \sigma \rangle}$ sending $H_{\mathbf{i}}$ to $\sum_\mu H_{(\mathbf{i},\mu)}$. Therefore we can combine the two homomorphisms

$$\mathfrak{g}(\Gamma)' \to (\mathfrak{g}')^{\langle \tilde{\mathbf{a}} \rangle} \qquad \text{and} \qquad \mathfrak{h}(\Gamma) \to M^{\langle \sigma \rangle}$$

to give a homomorphism

$$\mathfrak{g}(\Gamma) \to \mathfrak{g}^{\langle \sigma \rangle}.$$

This is a monomorphism by Proposition 1.4.3, and *a fortiori* $\mathfrak{g}(\Gamma)' \to (\mathfrak{g}')^{\langle \tilde{\mathbf{a}} \rangle}$ is a monomorphism.

■

Given σ as in the theorem, we define

$$\mathcal{L} := \mathfrak{g}^{\langle \sigma \rangle} = \mathfrak{n}_-^{\langle \tilde{\mathbf{a}} \rangle} \oplus M^{\langle \sigma \rangle} \oplus \mathfrak{n}_+^{\langle \tilde{\mathbf{a}} \rangle}.$$

This is a Lie algebra, with bracket product induced from that on $\mathfrak{g}$. We define a new bilinear form on $\mathcal{L}$ by $(-,-)_{\mathcal{L}} := \frac{1}{n}(-,-)_{\widetilde{\mathcal{Q}}}$. Then, by Lemma 7.1.1,

$$(H_{\mathbf{i}}, H_{\mathbf{j}})_{\mathcal{L}} = (H_{\mathbf{i}}, H_{\mathbf{j}})_{\Gamma}.$$

Therefore $(-,-)_{\mathcal{L}}$ and $(-,-)_{\Gamma}$ agree up to an automorphism of $\mathfrak{g}(\Gamma)$ fixing $\mathfrak{g}(\Gamma)'$. In particular, we may assume that the monomorphism

$$\mathfrak{g}(\Gamma) \hookrightarrow \mathcal{L}$$

preserves the bilinear form on $\mathfrak{h}(\Gamma)$ — and so we can identify $\mathfrak{g}(\Gamma)$ with its image in $\mathcal{L}$.

7.2 Generalised Kac-Moody Lie Algebras

The following is proved in [3].

Theorem 7.2.1. *Let $\mathfrak{g}$ be a generalised Kac-Moody Lie algebra with non-degenerate bilinear form on $\mathfrak{h}$. Then the subalgebra of $\mathfrak{g}$ fixed by a finite group of diagram automorphisms is again a generalised Kac-Moody Lie algebra with a non-degenerate bilinear form on its Cartan subalgebra.*

In particular, we see that $\mathcal{L}$ is a generalised Kac-Moody Lie algebra. Therefore we can define its set of roots and the root spaces. In general, however, there will be countably many imaginary simple roots (c.f. Section 7.4 discussing the affine Kac-Moody Lie algebras).

For convenience, we recall Borcherds' original definition of a generalised Kac-Moody Lie algebra [3].

Let $\mathcal{I}$ be a finite or countable set and $A = (a_{ij})$ a real symmetric matrix indexed by $\mathcal{I}$ satisfying

$$a_{ij} \leq 0 \quad \text{if } i \neq j, \qquad 2a_{ij}/a_{ii} \in \mathbb{Z} \quad \text{if } a_{ii} > 0. \tag{7.2.1}$$

Let $\mathfrak{h}$ be a complex vector space with a symmetric bilinear form $(-,-)$ and containing elements H_i for $i \in \mathcal{I}$ such that $(H_i, H_j) = a_{ij}$. N.B. The elements H_i are not assumed to be linearly independent.

Then the generalised Kac-Moody Lie algebra $\mathfrak{g}$ associated to A (and $\mathfrak{h}$) is generated by $\mathfrak{h}$ and elements E_i, F_i for $i \in \mathcal{I}$ subject to the following relations:

$$\begin{aligned}
&[H, H'] = 0, \\
&[E_i, F_j] = \delta_{ij} H_i, \\
&[H, E_j] = (H, H_j) E_j \quad \text{and} \quad [H, F_j] = -(H, H_j) F_j, \\
&(\operatorname{ad} E_i)^{1-2a_{ij}/a_{ii}} E_j = 0 \quad \text{and} \quad (\operatorname{ad} F_i)^{1-2a_{ij}/a_{ii}} F_j = 0 \quad \text{if } a_{ii} > 0, \\
&[E_i, E_j] = 0 \quad \text{and} \quad [F_i, F_j] = 0 \quad \text{if } a_{ij} = 0.
\end{aligned} \tag{7.2.2}$$

We can again define the roots of $\mathfrak{g}$, but these are not in general elements of the dual space $\mathfrak{h}^*$. Instead, we define the root lattice $\mathbb{Z}\mathcal{I}$ to be the free abelian group on elements e_i for $i \in \mathcal{I}$. We give $\mathbb{Z}\mathcal{I}$ a bilinear form $(-,-)$ by setting $(e_i, e_j) = a_{ij}$.

We have a grading of $\mathfrak{g}$ by $\mathbb{Z}\mathcal{I}$. This is given by putting $\deg E_i = e_i$, $\deg F_i = -e_i$ and $\deg H = 0$ for all $H \in \mathfrak{h}$. The roots of $\mathfrak{g}$ are now defined to be those α such that $\mathfrak{g}_\alpha \neq 0$. The e_i are called the simple roots and we say that a root α is real if $(\alpha, \alpha) > 0$ and imaginary otherwise. We note that we can have imaginary simple roots.

N.B. Every symmetrisable Kac-Moody is a generalised Kac-Moody Lie algebra. To see this, let $C = D^{-1}B$ be a symmetrisable GCM and let E_i, F_i and H_i be the Chevalley generators for the Kac-Moody Lie algebra $\mathfrak{g}(C)$. Then, choosing $\varepsilon_i^2 = d_i$, the generators $\widehat{E}_i = \varepsilon_i E_i$, $\widehat{F}_i = \varepsilon_i F_i$ and $\widehat{H}_i = d_i H_i$ satisfy the relations for the generalised Kac-Moody Lie algebra with matrix B.

7.3 The Structure of $\mathcal{L}$

In this section we describe an alternative way of viewing $\mathcal{L}$, namely as an integrable $\mathfrak{g}(\Gamma)$-module. We show that the non-zero weights of $\mathcal{L}$ are precisely the roots of $\mathfrak{g}(\Gamma)$ and that the real roots all have multiplicity one.

We keep the notation of Section 7.1.

Recall from Section 3.2 the map $h : \mathbb{Z}\widetilde{\mathcal{I}} \to \mathbb{Z}\mathbf{I}$ such that $h(\beta)_{\mathbf{i}} = \sum_\mu \beta_{(\mathbf{i},\mu)}$. By Lemma 3.6.1 this satisfies

$$\big(\sum_\mu e_{(\mathbf{i},\mu)}, \beta\big)_{\widetilde{\mathcal{Q}}} = \frac{1}{d_{\mathbf{i}}}(e_{\mathbf{i}}, h(\beta))_\Gamma$$

and h maps $\Delta(\widetilde{\mathcal{Q}})$ onto $\Delta(\Gamma)$ by Proposition 3.5.4.

Proposition 7.3.1. *The adjoint action of* $\mathfrak{g}(\Gamma)$ *on* $\mathcal{L}$ *is* $\mathfrak{h}(\Gamma)$*-diagonalisable and the non-zero weights are precisely the roots of* Γ*.*

Proof.
Let $\beta \in \Delta(\widetilde{\mathcal{Q}})$ be of order m under $\tilde{\mathbf{a}}$, so $\tilde{\mathbf{a}}^m$ acts on the root space $\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta$. We first note that, since we have identified the realisations $(\mathfrak{h}(\Gamma), \{H_{\mathbf{i}}\}, \{e_{\mathbf{i}}\})$ and $(M^{\langle\sigma\rangle}, \{H_{\mathbf{i}}\}, \{e_{\mathbf{i}}\})$, the notation $e_{\mathbf{i}}(H)$ is unambiguous. Also, for any $H \in M^{\langle\sigma\rangle}$,

$$e_{(\mathbf{i},\mu)}(H) = \frac{d_{\mathbf{i}}}{n}\sum_\mu e_{(\mathbf{i},\mu)}(H) = e_{\mathbf{i}}(H).$$

Therefore, writing $\beta = \sum_{(\mathbf{i},\mu)} \beta_{(\mathbf{i},\mu)} e_{(\mathbf{i},\mu)}$, we have

$$\beta(H) = \sum_{(\mathbf{i},\mu)} \beta_{(\mathbf{i},\mu)} e_{(\mathbf{i},\mu)}(H) = \sum_{\mathbf{i}} \big(\sum_\mu \beta_{(\mathbf{i},\mu)}\big) e_{\mathbf{i}}(H) = \sum_{\mathbf{i}} h(\beta)_{\mathbf{i}} e_{\mathbf{i}}(H) = h(\beta)(H).$$

Now suppose that $x \in \mathfrak{g}(\widetilde{\mathcal{Q}})_\beta$ satisfies $\tilde{\mathbf{a}}^m(x) = x$ and let $X = x + \tilde{\mathbf{a}}(x) + \cdots + \tilde{\mathbf{a}}^{m-1}(x) \in \mathcal{L}$. As $h(\tilde{\mathbf{a}}^r(\beta)) = h(\beta)$ for all r, we see that

$$[H, X] = \sum_r [H, \tilde{\mathbf{a}}^r(x)] = \sum_r \tilde{\mathbf{a}}^r(\beta)(H)\tilde{\mathbf{a}}^r(x) = h(\beta)(H)\sum_r \tilde{\mathbf{a}}^r(x) = h(\beta)(H)X.$$

Therefore X lies in the weight space $\mathcal{L}_{h(\beta)}$.

Since any element in $\mathcal{L}$ can be expressed as a sum of such orbit sums of homogeneous elements, we deduce that $\mathcal{L}$ is $\mathfrak{h}(\Gamma)$-diagonalisable. Finally, since h maps $\Delta(\widetilde{\mathcal{Q}})$ onto $\Delta(\Gamma)$, the non-zero weights must all be roots of Γ. Conversely, $\mathfrak{g}(\Gamma) \subset \mathcal{L}$ so every root appears as a weight.
■

Corollary 7.3.2. *The monomorphism $\mathfrak{g}(\Gamma) \hookrightarrow \mathcal{L}$ endows $\mathcal{L}$ with the structure of an integrable $\mathfrak{g}(\Gamma)$-module.*

Proof.
We have shown that $\mathcal{L}$ is an $\mathfrak{h}(\Gamma)$-diagonalisable module and that the weights are precisely the roots of Γ. Also, for any root $\alpha \in \Delta(\Gamma)$, the set

$$\{\alpha + ke_{\mathbf{i}} \mid k \in \mathbb{Z}\} \cap \Delta(\Gamma)$$

is finite ([25], Proposition 3.6). Therefore $E_{\mathbf{i}}$ and $F_{\mathbf{i}}$ must act locally nilpotently on $\mathcal{L}$.
■

For any root $\beta \in \Delta(\widetilde{\mathcal{Q}})$, say of order m under $\tilde{\mathbf{a}}$, we denote by $E(\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta, 1)$ the eigenspace with eigenvalue 1 for the action of $\tilde{\mathbf{a}}^m$ on $\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta$.

Corollary 7.3.3. *The weight space $\mathcal{L}_\alpha$ for $\alpha \in \Delta(\Gamma)$ has dimension*

$$\sum_{[\beta] \in h^{-1}(\alpha)} \dim E(\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta, 1),$$

where we are summing over $\tilde{\mathbf{a}}$-orbits of roots β such that $h(\beta) = \alpha$.

Proof.
It follows from the proof of the proposition that $\mathcal{L}_\alpha$ is spanned by elements of the form $X = x + \cdots + \tilde{\mathbf{a}}^{m-1}(x)$, where $x \in E(\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta, 1)$, $h(\beta) = \alpha$ and β has order m under $\tilde{\mathbf{a}}$. Clearly it is enough to take one representative β from each $\tilde{\mathbf{a}}$-orbit, and elements coming from distinct $\tilde{\mathbf{a}}$-orbits must be linearly independent.
■

Corresponding to the reflection $\tilde{r}_{(\mathbf{i},\mu)} \in W(\Gamma)$ we have the automorphism $\exp(\operatorname{ad} F_{(\mathbf{i},\mu)}) \exp(-\operatorname{ad} E_{(\mathbf{i},\mu)}) \exp(\operatorname{ad} F_{(\mathbf{i},\mu)})$ of $\mathfrak{g}(\widetilde{\mathcal{Q}})$. We shall abuse notation and denote this by $\tilde{r}_{(\mathbf{i},\mu)}$ as well. This satisfies (Lemma 3.8 in [25])

1. $\tilde{r}_{(\mathbf{i},\mu)}(\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta) = \mathfrak{g}(\widetilde{\mathcal{Q}})_{\tilde{r}_{(\mathbf{i},\mu)}(\beta)}$;
2. $\tilde{r}_{(\mathbf{i},\mu)}(H) = H - e_{(\mathbf{i},\mu)}(H)H_{(\mathbf{i},\mu)}$ for all $H \in \mathfrak{h}(\widetilde{\mathcal{Q}})$.

Since the automorphisms $\tilde{r}_{(\mathbf{i},\mu)}$ and $\tilde{r}_{(\mathbf{i},\nu)}$ of $\mathfrak{g}(\widetilde{\mathcal{Q}})$ commute, we can define the automorphism $\tilde{s}_{\mathbf{i}} = \prod_{\mu} \tilde{r}_{(\mathbf{i},\mu)}$. Also, $\tilde{s}_{\mathbf{i}}$ and σ commute on $\mathfrak{n}_- \oplus (M + \mathfrak{h}') \oplus \mathfrak{n}_+$. For, $\tilde{s}_{\mathbf{i}}$ and $\tilde{\mathbf{a}}$ clearly commute on $\mathfrak{g}(\widetilde{\mathcal{Q}})'$, so we just need to consider their actions on M. As in Proposition 7.1.4, we can choose $Y_1, \ldots, Y_s \in M^{\langle\sigma\rangle}$ spanning a subspace complementary to $M \cap \mathfrak{h}(\widetilde{\mathcal{Q}})'$ in M. Then

$$\tilde{s}_{\mathbf{i}}(Y_j) = Y_j - \sum_{\mu} e_{(\mathbf{i},\mu)}(Y_j) H_{(\mathbf{i},\mu)} = Y_j - e_{\mathbf{i}}(Y_j) \sum_{\mu} H_{(\mathbf{i},\mu)} = Y_j - e_{\mathbf{i}}(Y_j) H_{\mathbf{i}},$$

which is again fixed by σ.

We deduce that $\tilde{s}_{\mathbf{i}}$ defines an automorphism of $\mathcal{L}$ such that

1. $\tilde{s}_{\mathbf{i}}(\mathcal{L}_\alpha) = \mathcal{L}_{\tilde{s}_{\mathbf{i}}(\alpha)}$;
2. $\tilde{s}_{\mathbf{i}}(H) = H - e_{\mathbf{i}}(H) H_{\mathbf{i}}$ for all $H \in M^{\langle\sigma\rangle} = \mathfrak{h}(\Gamma)$.

It follows that $\tilde{s}_{\mathbf{i}}$ extends the natural action of $\exp(\operatorname{ad} F_{\mathbf{i}}) \exp(-\operatorname{ad} E_{\mathbf{i}}) \exp(\operatorname{ad} F_{\mathbf{i}})$ on $\mathfrak{g}(\Gamma)$ to the whole of $\mathcal{L}$.

Proposition 7.3.4. *Let $\beta \in \Delta(\widetilde{\mathcal{Q}})^{\mathrm{re}}$ be of order m under $\tilde{\mathbf{a}}$. Then $\tilde{\mathbf{a}}^m$ acts as the identity on $\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta$. In particular,* $\dim \mathcal{L}_\alpha = 1$ *for all $\alpha \in \Delta(\Gamma)^{\mathrm{re}}$.*

Proof.
By definition of how $\tilde{\mathbf{a}}$ acts on the Chevalley generators, we know the result for all simple roots β.

In general, let $\beta \in \Delta(\widetilde{\mathcal{Q}})$ be of order m under $\tilde{\mathbf{a}}$ such that $h(\beta)$ is a real root of Γ. By Lemma 3.6.1 there exists $w \in C_{\tilde{\mathbf{a}}}(\widetilde{\mathcal{Q}})$ such that $w(\beta)$ is simple and has order m under $\tilde{\mathbf{a}}$.

Suppose that $w = \tilde{s}_{\mathbf{i}_1} \cdots \tilde{s}_{\mathbf{i}_r}$ and consider the corresponding element $w = \tilde{s}_{\mathbf{i}_1} \cdots \tilde{s}_{\mathbf{i}_r} \in \operatorname{Aut}(\mathfrak{g}(\widetilde{\mathcal{Q}}))$. Then $\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta = w^{-1}(\mathfrak{g}(\widetilde{\mathcal{Q}})_{w(\beta)})$ and hence is fixed by $\tilde{\mathbf{a}}^m$.

Now let $\alpha \in \Delta(\Gamma)^{\mathrm{re}}$. By Proposition 3.6.2 there is a unique $\tilde{\mathbf{a}}$-orbit of roots mapping to α under h, all of which are real. Let β be one such root. Then

$\dim E(\mathfrak{g}(\widetilde{\mathcal{Q}})_\beta, 1) = 1$ and so $\dim \mathcal{L}_\alpha = 1$ by Corollary 7.3.3.

■

We can now generalise Kac's conjecture on the number of isomorphism classes of absolutely indecomposable representations over a finite field.

Conjecture 2'. For $\alpha \in \Delta(\Gamma)_+$ the constant term $A\text{-}ii(\alpha, 0)$ equals the weight multiplicity $\dim \mathcal{L}_\alpha$.

We have seen that the conjecture holds for all positive real roots, since both terms equal 1. In particular, it holds whenever Γ is Dynkin. In the next section we prove that the conjecture holds whenever Γ is affine.

We note that if $\widetilde{\mathcal{Q}}$ is Dynkin, then $\mathfrak{g}(\widetilde{\mathcal{Q}})$ is simple and $\mathcal{L} = \mathfrak{g}(\widetilde{\mathcal{Q}})^{\langle \tilde{\mathbf{a}} \rangle}$. As all roots for Γ are real, the above proposition implies that

$$\mathfrak{g}(\Gamma) \cong \mathfrak{g}(\widetilde{\mathcal{Q}})^{\langle \tilde{\mathbf{a}} \rangle}.$$

This is the classical realisation of the non-symmetric finite dimensional simple Lie algebras, as in Proposition 7.9 of [25].

7.4 The Affine Kac-Moody Lie Algebras

Let $\mathfrak{g}$ be a symmetric affine Kac-Moody Lie algebra. There is a concrete description of $\mathfrak{g}$ in terms of its 'underlying' finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$ (see Chapters 6 and 7 of [25]). Here we shall describe the construction of the derived Lie algebra $\mathfrak{g}'$ and then use this to calculate the dimensions $\dim E(\mathfrak{g}_{m\delta}, 1)$, where δ is the minimal positive imaginary root.

Let $\widetilde{\mathcal{Q}}$ be a connected affine quiver with GCM $\widetilde{A}$, $\mathfrak{g}$ the corresponding symmetric affine Lie algebra and $\widetilde{\mathcal{I}} = \{v, 1, \ldots, l\}$, where l is the rank of $\widetilde{A}$ and v is an exending vertex of $\widetilde{\mathcal{Q}}$ — that is, if we remove v from $\widetilde{\mathcal{Q}}$ then the resulting quiver is connected Dynkin. The subalgebra $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$ generated by the E_i and F_i for

$i = 1, \dots, l$ is simple and finite dimesional and we have the following description of the derived subalgebra $\mathfrak{g}'$.

$$\mathfrak{g}' \cong \mathbb{C}[t, t^{-1}] \otimes \mathring{\mathfrak{g}} \oplus \mathbb{C}c$$

with bracket product

$$[t^m \otimes x + \lambda c, t^n \otimes y + \mu c] := t^{m+n} \otimes [x, y] + m\delta_{m,-n}(x, y)c.$$

We identify $\mathring{\mathfrak{g}}$ with its image $1 \otimes \mathring{\mathfrak{g}}$.

Write $\theta = \sum_i \theta_i e_i \in \Delta(\mathring{\mathfrak{g}})_+$ for the highest root (i.e. $\theta + e_i \notin \Delta(\mathring{\mathfrak{g}})$ for any i) and put $H_\theta = \sum_i \theta_i H_i \in \mathring{\mathfrak{h}}$. Let $\mathring{\omega}$ be the Chevalley involution of $\mathring{\mathfrak{g}}$ and pick $E_\theta \in \mathring{\mathfrak{g}}_\theta$ such that $(E_\theta, \mathring{\omega}(E_\theta)) = -1$. Set $F_\theta = -\mathring{\omega}(E_\theta) \in \mathring{\mathfrak{g}}_{-\theta}$ so that $[E_\theta, F_\theta] = H_\theta$.

Define $E_v := t \otimes F_\theta$, $F_v := t^{-1} \otimes E_\theta$ and $H_v := c - H_\theta$. Then the Chevalley generators for $\mathfrak{g}$ are

$$E_v, E_1, \dots, E_l, \qquad F_v, F_1, \dots, F_l, \qquad H_v, H_1, \dots, H_l$$

and the simple roots are $e_v := \delta - \theta, e_1, \dots, e_l$.

We have (Proposition 6.3 and Corollary 7.4 of [25])

Proposition 7.4.1. *The roots of* $\mathfrak{g}$ *are*

$$\Delta^{\mathrm{re}} = \{\alpha + m\delta \mid \alpha \in \Delta(\mathring{\mathfrak{g}}), m \in \mathbb{Z}\}, \qquad \Delta^{\mathrm{im}} = \{m\delta \mid 0 \neq m \in \mathbb{Z}\}.$$

Moreover, the root space $\mathfrak{g}_{m\delta}$ *has basis* $t^m \otimes H_i$ *for* $i = 1, \dots, l$.

As mentioned in the previous chapter, every symmetrisable affine Kac-Moody Lie algebra also has a tier number t, which is the smallest positive integer such that

$$\Delta^{\mathrm{re}} + t\delta = \Delta^{\mathrm{re}}.$$

Now let $(\mathcal{Q}, \mathbf{a})$ be a quiver with automorphism, Γ the associated valued graph and $(\widetilde{\mathcal{Q}}, \tilde{\mathbf{a}})$ the dual quiver with automorphism. We note that $\tilde{\mathbf{a}}(\delta) = \delta$ since it is the unique minimal positive imaginary root. Let δ_Γ be the minimal positive imaginary root of Γ. Then $h(\delta) = r\delta_\Gamma$ for some positive integer r.

N.B. In all cases except Case 1 of Chapter 6 r is the tier number of Γ, whereas in Case 1, the tier number is 1 but if t is the order of $\mathbf{a}$ on $\mathcal{I}$, then $r = n/t$.

We now describe how $\tilde{\mathbf{a}}$ acts on the imaginary root spaces $\mathfrak{g}_{m\delta}$ $(0 \neq m \in \mathbb{Z})$.

Theorem 7.4.2. *Set* $K_i^{(m)} := t^m \otimes H_i$ *for* $i = 1, \ldots, l$ *and* $K_v^{(m)} := -t^m \otimes H_\theta$. *Then* $K_1^{(m)}, \ldots, K_l^{(m)}$ *give a basis for* $\mathfrak{g}_{m\delta}$ *and* $\tilde{\mathbf{a}} : K_i^{(m)} \mapsto K_{\tilde{\mathbf{a}}(i)}^{(m)}$ *for all* $i \in \widetilde{\mathcal{I}}$.

To prove this, we consider the two different cases — whether or not $\widetilde{\mathcal{Q}}$ has a fixed vertex.

Case 1.

Suppose that $\widetilde{\mathcal{Q}}$ does not have a fixed vertex. Then $\widetilde{\mathcal{Q}}$ is of type $\widetilde{\mathbb{A}}_l$ with $l = ts-1$ and $\tilde{\mathbf{a}}$ is rotation by $2\pi/t$. That is, we can label the vertices clockwise from $v = 0$ to l and $\tilde{\mathbf{a}}$ acts by $i \mapsto i + s$. We note that $\mathring{\mathfrak{g}}$ is of type $\mathbb{A}_l$ with vertices from 1 to l.

For $1 \leq i \leq j \leq l$, set $E_{[1,1]} := E_1$ and $E_{[i,j]} := [E_i E_{i+1} \cdots E_j]$, where we are using the convention that $[xy \cdots z] := [x, [y, [\cdots z]]]$. Then

$$\mathring{\omega}(E_{[i,j]}) = (-1)^{j-i+1}[F_i \cdots F_j] = -[F_j \cdots F_i] =: -F_{[j,i]}.$$

For, by induction on $j - i$,

$$F_{[i,j]} = (-1)^{j-i-1}[F_i, F_{[j,i+1]}] = (-1)^{j-i-1}[F_j, [\cdots [F_i, F_{i+1}]]] = (-1)^{j-i} F_{[j,i]}$$

Also,

$$(E_{[i,j]}, \mathring{\omega}(E_{[i,j]})) = -(E_{[i,j]}, F_{[j,i]}) = -1.$$

For, $(E_i, F_i) = 1$ and if $i < j$, then $[E_i, F_{[i,j]}] = [H_i F_{i+1} \cdots F_j] = F_{[i+1,j]}$. Therefore by induction,

$$(E_{[i,j]}, F_{[i,j]}) = -(E_{[i+1,j]}, [E_i F_{[i,j]}]) = -(E_{[i+1,j]}, F_{[i+1,j]}) = (-1)^{j-i}$$

In particular, $E_\theta = E_{[1,l]}$ and $F_\theta = F_{[l,1]}$.

We now wish to calculate $[F_{[k,i]}, E_{[i,j]}]$ for $1 \leq i \leq j < k \leq l$. We first note that

$$\operatorname{ad} F_k \cdots \operatorname{ad} F_{j+1} [E_{[i,j]}, F_{[j,i]}] = [E_{[i,j]}, F_{[k,i]}]$$

and also

$$[E_{[i,j]}, F_{[j,i]}] = (E_{[i,j]}, F_{[j,i]})(H_i + \cdots + H_j) = H_i + \cdots + H_j.$$

Hence

$$\begin{aligned}
[F_{[k,i]}, E_{[i,j]}] &= -\operatorname{ad} F_k \cdots \operatorname{ad} F_{j+1}(H_i + \cdots + H_j) \\
&= -\operatorname{ad} F_k \cdots \operatorname{ad} F_{j+1}(H_j) = [F_k \cdots F_{j+1}] = F_{[k,j+1]}.
\end{aligned}$$

Since $[E_j F_{[j,i]}] = F_{[j-1,i]}$ for all $j > i$, we have that

$$\begin{aligned}
[E_1 \cdots E_l E_0] &= t \otimes [E_1 \cdots E_l F_{[l,1]}] = t \otimes [E_1, F_1] = t \otimes H_1 = K_1^{(1)}; \\
[E_i \cdots E_l E_0 E_{[1,i-1]}] &= t \otimes [E_i \cdots E_l F_{[l,1]} E_{[1,i-1]}] = t \otimes [E_i \cdots E_l F_{[l,i]}] \\
&= t \otimes H_i = K_i^{(1)} \qquad \text{for } 1 < i \leq l; \\
[E_0 E_{[1,l]}] &= t \otimes [F_{[l,1]} E_{[1,l]}] = -t \otimes (H_1 + \cdots + H_l) = K_0^{(1)}.
\end{aligned}$$

We deduce that $\tilde{\mathbf{a}}(K_i^{(1)}) = K_{\tilde{\mathbf{a}}(i)}^{(1)}$.

Finally, let us abuse notation slightly and write $t^m \otimes E_0 := t^{m+1} \otimes F_\theta$. Then for all $0 \leq i \leq l$ we have that $2t^m \otimes E_i = [K_i^{(m)}, E_i]$, and similarly for F. Also,

$$K_i^{(m+1)} = [t^m \otimes E_i, t \otimes F_i]$$

and so by induction on m we get that $\tilde{\mathbf{a}}(K_i^{(m)}) = K_{\tilde{\mathbf{a}}(i)}^{(m)}$ for all m.

Case 2.

Now suppose that $\widetilde{\mathcal{Q}}$ has a fixed vertex ε. We note that we can always take $1 \leq \varepsilon \leq l$. Then $t^m \otimes E_\varepsilon$ is fixed by $\tilde{\mathbf{a}}$ since it is in the root space corresponding to $m\delta + e_\varepsilon$, which is a real root fixed by $\tilde{\mathbf{a}}$. Hence $K_\varepsilon^{(m)} = [t^m \otimes E_\varepsilon, F_\varepsilon]$ is also fixed by $\tilde{\mathbf{a}}$.

Let $1 \leq i \leq l$ be a vertex such that $\tilde{\mathbf{a}}(K_i^{(m)}) = K_{\tilde{\mathbf{a}}(i)}^{(m)}$ and take $1 \leq j \leq l$ adjacent to i (i.e. $\tilde{a}_{ij} \neq 0$). Then $[E_j F_j K_i^{(m)}] = \tilde{a}_{ij} K_j^{(m)}$ and so

$$\tilde{a}_{ij}\tilde{\mathbf{a}}(K_j^{(m)}) = [E_{\tilde{\mathbf{a}}(j)} F_{\tilde{\mathbf{a}}(j)} K_{\tilde{\mathbf{a}}(i)}^{(m)}] = \tilde{a}_{\tilde{\mathbf{a}}(i)\tilde{\mathbf{a}}(j)} K_{\tilde{\mathbf{a}}(j)}^{(m)} = \tilde{a}_{ij} K_{\tilde{\mathbf{a}}(j)}^{(m)}.$$

Therefore, since $\widetilde{\mathcal{Q}}$ is connected, $\tilde{\mathbf{a}}(K_i^{(m)}) = K_{\tilde{\mathbf{a}}(i)}^{(m)}$ for all i such that $i \neq v \neq \tilde{\mathbf{a}}(i)$.

Now, either v is also a fixed vertex, in which case $\tilde{\mathbf{a}}(K_v^{(m)}) = K_v^{(m)}$ by the linear dependence of the $K_i^{(m)}$, or $\tilde{\mathbf{a}}(j) = v$ for some $1 \leq j \leq l$. In this case, take i adjacent to j (so i and v will be in different $\tilde{\mathbf{a}}$-orbits). Then

$$\tilde{a}_{ij}\tilde{\mathbf{a}}(K_j^{(m)}) = [E_v F_v K_{\tilde{\mathbf{a}}(i)}^{(m)}] = [t \otimes F_\theta, [t^{-1} \otimes E_\theta, t^m \otimes H_{\tilde{\mathbf{a}}(i)}]].$$

Using that $\delta = e_v + \theta$ and $\tilde{\mathbf{a}} \in \mathrm{Aut}(\widetilde{A})$, we have

$$[H_{\tilde{\mathbf{a}}(i)}, E_\theta] = (e_{\tilde{\mathbf{a}}(i)}, \theta) E_\theta = -(e_{\tilde{\mathbf{a}}(i)}, e_v) E_\theta = -\tilde{a}_{\tilde{\mathbf{a}}(i)v} E_\theta = -\tilde{a}_{ij} E_\theta.$$

Thus

$$\tilde{a}_{ij}\tilde{\mathbf{a}}(K_j^{(m)}) = \tilde{a}_{ij} t^m \otimes [F_\theta, E_\theta] = -\tilde{a}_{ij} t^m \otimes H_\theta = \tilde{a}_{ij} K_v^{(m)}.$$

Therefore we again see that $\tilde{\mathbf{a}}(K_i^{(m)}) = K_{\tilde{\mathbf{a}}(i)}^{(m)}$ for all i.

This proves Theorem 7.4.2.

Proposition 7.4.3. *Let r be given by $h(\delta) = r\delta_\Gamma$. Then $\mathcal{L}_{mr\delta_\Gamma} = \mathfrak{g}(\Gamma)_{mr\delta_\Gamma}$ for all $m \neq 0$ and both spaces have dimension $|\mathbf{I}| - 1$.*

Proof.

We have shown that $E(\mathfrak{g}_{m\delta}, 1)$ is spanned by $\sum_\mu K_{(\mathbf{i},\mu)}^{(m)}$ and we know that these elements satisfy a linear equation. Therefore $\dim E(\mathfrak{g}_{m\delta}, 1) = |\mathbf{I}| - 1$. Conversely, the only root of $\mathfrak{g}$ mapping to $mr\delta_\Gamma$ is $m\delta$. For, every root is of the form $s\delta \pm \alpha$ with either $\alpha = 0$ and $s \neq 0$ or $\alpha \in \Delta(\mathring{\mathfrak{g}})$ and any $s \in \mathbb{Z}$. Also, for all $\alpha > 0$,

$$1 \leq \mathrm{ht}\,\alpha \leq \mathrm{ht}\,\theta = \mathrm{ht}\,\delta - 1 = r\mathrm{ht}\,\delta_\Gamma - 1.$$

Thus the proposition follows by Corollary 7.3.3.

■

We can now do a case-by-case analysis to calculate which roots of $\mathfrak{g}$ map to $m\delta_\Gamma$ for $m \not\equiv 0 \bmod r$. We summarise this as

Proposition 7.4.4. *Let r be given by $h(\delta) = r\delta_\Gamma$. Then*

$$\dim \mathcal{L}_{m\delta_\Gamma} - \dim \mathfrak{g}(\Gamma)_{m\delta_\Gamma} = \begin{cases} 1 & m \not\equiv 0 \bmod r; \\ 0 & m \equiv 0 \bmod r. \end{cases}$$

Combining this with Theorem 6.3.1 we conclude that

Theorem 7.4.5. *Let r be given by $h(\delta) = r\delta_\Gamma$. Then for all $\alpha \in \Delta(\Gamma)_+$*

$$A\text{-}ii(\alpha, 0) = \dim \mathcal{L}_\alpha.$$

Thus we have proved Conjecture $2'$ when $\mathcal{Q}$ is affine.

Chapter 8

Representations of Species

In this chapter we will study the connections between ii-representations and representations of species over finite fields. In particular, we will associate to each valued graph Γ a quiver with an automorphism $(\mathcal{Q}, \mathbf{a})$ such that the corresponding valued graph of the pair $(\mathcal{Q}, \mathbf{a})$ is precisely the underlying graph of Γ. Then for L a finite field and M an extension of degree t, the order of $\mathbf{a}$, there is a natural action of the Galois group $\mathrm{Gal}(M/L)$ on $\mathrm{Rep}(\mathcal{Q}, M)$ via the Frobenius automorphism (c.f. Chapter 4). We show that there is a bijection between the isomorphism classes of representations of the L-species S of Γ and those representations of $\mathcal{Q}$ over M for which the actions of $\mathbf{a}$ and the Galois group coincide.

In particular, we observe that every ii-indecomposable for $(\mathcal{Q}, \mathbf{a})$ defined over L must correspond to an indecomposable for S. Using this, we show that the polynomials giving the number of isomorphism classes of indecomposables for S of dimension vector α are non-zero if and only if α is a positive root for Γ. This completes the analogue of Kac's Theorem for L-species [7, 8, 18].

8.1 Species over Finite Fields

Let Γ be a valued quiver with symmetrisable GCM $C = D^{-1}B$. We fix a finite field $L = \mathbb{F}_q$ with algebraic closure K and denote by $\mathbb{F}_{q^r}$ the unique extension of

L of degree r inside K. The L-species S of Γ is given by the field $\mathbb{F}_{q^{d_i}}$ at vertex i and the $\mathbb{F}_{q^{d_j}}$-$\mathbb{F}_{q^{d_i}}$-bimodule $\mathbb{F}_{q^{b_{ij}}}$ for each arrow $i \to j$ in Γ.

A representation X of S is given by an $\mathbb{F}_{q^{d_i}}$-vector space X_i for each vertex i and an $\mathbb{F}_{q^{d_j}}$-linear map

$$\theta_{ij} : \mathbb{F}_{q^{b_{ij}}} \otimes_{\mathbb{F}_{q^{d_i}}} X_i \to X_j \tag{8.1.1}$$

for each arrow $i \to j$. Therefore, the category of S-representations is equivalent to the category of finite dimensional modules for the tensor algebra $\Lambda := T(\Lambda_0, \Lambda_1)$, where $\Lambda_0 = \prod_i \mathbb{F}_{q^{d_i}}$ is a semisimple algebra and $\Lambda_1 = \coprod_{i \to j} \mathbb{F}_{q^{b_{ij}}}$ is a Λ_0-bimodule.

In [8], Dlab and Ringel studied the representations of a L-species associated to a valued affine quiver Γ. This was later extended by Hua [18] for general valued quivers (without oriented cycles) and a partial analogue of Kac's Theorem was proved. This was completed by Deng and Xiao [7] using the Ringel-Hall algebra.

Theorem 8.1.1 (Hua, Deng-Xiao). *Let Γ be a valued quiver, L a finite field and S the L-species of Γ. Then*

1. *the dimension vectors of the indecomposable S-representatons are precisely the positive roots of the symmetrisable Kac-Moody Lie algebra $\mathfrak{g}(\Gamma)$;*
2. *if α is real, there is a unique (up to isomorphism) indecomposable of dimension vector α.*

Let M/L be a splitting field for each $\mathbb{F}_{q^{d_i}}$ — for example, we can take $M = \mathbb{F}_{q^t}$, where $t = \mathrm{lcm}\{d_i\}$. Then the algebra $M \otimes_L \Lambda$ is isomorphic to the path algebra of a quiver $\mathcal{Q}$. Consider the action of the Galois group $G := \mathrm{Gal}(M/L)$ on $M \otimes_L \Lambda$ generated by $g : a \otimes b \mapsto a^q \otimes b$. It follows from Corollary 4.1.4 that an $M \otimes_L \Lambda$-module is induced up from a Λ-module if and only if it is invariant under this action. We show that this action can be thought of as an admissible quiver automorphism combined with the G-action on $M\mathcal{Q}$ coming from the identification $M\mathcal{Q} \cong M \otimes_L L\mathcal{Q}$.

We describe the situation in the case when Γ has only two vertices; the general case following immediately. Let Γ be the valued quiver

$$i \xrightarrow{(|c_{ji}|,|c_{ij}|)} j$$

with $d_i c_{ij} = b_{ij} = d_j c_{ji}$. Let t be the lowest common multiple of d_i and d_j and set $|b_{ij}| = bt$. The tensor algebra is therefore given by

$$\Lambda = \begin{pmatrix} \mathbb{F}_{q^{d_i}} & 0 \\ \mathbb{F}_{q^{bt}} & \mathbb{F}_{q^{d_j}} \end{pmatrix} \cong \begin{pmatrix} \mathbb{F}_{q^{d_i}} & 0 \\ M^b & \mathbb{F}_{q^{d_j}} \end{pmatrix}.$$

Now $M \otimes_L \Lambda \cong M\mathcal{Q}$ for some quiver $\mathcal{Q}$ and we can assume that $b = 1$ since for general b we just take b copies of each arrow in $\mathcal{Q}$.

By the Normal Basis Theorem [4], there exists $x \in \mathbb{F}_{q^{d_i}}$ such that the elements x^{q^r} for $0 \leq r < d_i$ form an L-basis for $\mathbb{F}_{q^{d_i}}$. We also have the L-algebra isomorphism

$$\mathbb{F}_{q^{d_i}} \otimes_L \mathbb{F}_{q^{d_i}} \xrightarrow{\sim} \prod_{\mu=0}^{d_i-1} \mathbb{F}_{q^{d_i}}, \qquad a \otimes b \mapsto (ab^{q^\mu})_\mu. \tag{8.1.2}$$

Let $\varepsilon_{(i,\mu)}$ denote the unit for the μ-th copy of $\mathbb{F}_{q^{d_i}}$ and fix $a_r \in \mathbb{F}_{q^{d_i}}$ such that $\sum_r a_r \otimes x^{q^r}$ corresponds to $\varepsilon_{(i,0)}$. Then $\sum_r a_r \otimes x^{q^{r-\mu}}$ corresponds to $\varepsilon_{(i,\mu)}$ and so

$$\sum_{r,s} a_r a_s \otimes x^{q^{r-\lambda}} x^{q^{s-\mu}} = \delta_{\lambda,\mu} \sum_r a_r \otimes x^{q^{r-\lambda}}. \tag{8.1.3}$$

Similarly we can find y and b_s in M such that the $y^{q^{sd_i}}$ give a basis for M over $\mathbb{F}_{q^{d_i}}$ and $\sum_s b_s y^{q^{(s-\mu)d_i}} = \delta_{\mu,0}$. Then, for the isomorphism

$$M \otimes_L M \xrightarrow{\sim} \prod_{\lambda=0}^{t-1} M, \qquad a \otimes b \mapsto (ab^{q^\lambda})_\lambda, \tag{8.1.4}$$

the λ-th unit, which we shall denote by $\rho_{(\lambda)}$, corresponds to $\sum_{r,s} a_r b_s \otimes x^{q^{r-\lambda}} y^{q^{sd_i-\lambda}}$. It follows that

$$\rho_{(\lambda)} \varepsilon_{(i,\mu)} = \begin{cases} \rho_{(\lambda)} & \text{if } \lambda \equiv \mu \bmod d_i; \\ 0 & \text{otherwise.} \end{cases} \tag{8.1.5}$$

We have analogous identities for the isomorphism $M \otimes_L \mathbb{F}_{q^{d_j}} \xrightarrow{\sim} \prod_{\nu=0}^{d_j-1} M$ and the product $\varepsilon_{(j,\nu)}\rho_{(\lambda)}$. In conclusion, we have

Lemma 8.1.2. *Let* Γ *be a valued quiver with symmetrisable GCM* $C = D^{-1}B$*,* L *a finite field and* Λ *the corresponding* L*-algebra. Let* t *be the lowest common multiple of the* d_i*. Then for* M/L *an extension of degree* t*, the algebra* $M \otimes_L \Lambda$ *is isomorphic to the path algebra of a quiver* $M\mathcal{Q}$*. The vertices of* $\mathcal{Q}$ *are labelled by pairs* (i,μ) *for* i *a vertex of* Γ *and* $0 \leq \mu < d_i$ *and there are* $b_{ij}/\mathrm{lcm}(d_i,d_j)$ *arrows* $(i,\mu) \to (j,\nu)$ *if and only if* $i \to j$ *in* Γ *and* $\mu \equiv \nu \mod \mathrm{hcf}(d_i,d_j)$*.*

The Galois group $G = \mathrm{Gal}(M/L)$ acts naturally on Λ as L-algebra automorphisms and so we can extend this to get M-algebra automorphisms of $M \otimes_L \Lambda$. Let $\mathbf{a}$ be a generator for G, acting as $\mathbf{a}(a \otimes b) := a \otimes b^q$. In terms of the path algebra $M\mathcal{Q}$, we get

$$\mathbf{a}(\varepsilon_{(i,\mu)}) = \mathbf{a}\big(\sum_r a_r \otimes x^{q^{r-\mu}}\big) = \sum_r a_r \otimes x^{q^{r+1-\mu}} = \varepsilon_{(i,\mu-1)}. \tag{8.1.6}$$

Similarly, $\mathbf{a}(\varepsilon_{(j,\nu)}) = \varepsilon_{(j,\nu-1)}$ and $\mathbf{a}(\rho_{(\lambda)}) = \rho_{(\lambda-1)}$, and hence $\mathbf{a}$ acts on $M\mathcal{Q}$ as an admissible quiver automorphism. We note that the valued graph associated to the pair $(\mathcal{Q},\mathbf{a})$ is precisely the underlying graph of the valued quiver Γ.

On the other hand, we have the standard action of G on $M\mathcal{Q}$ as L-algebra automorphisms, coming from the identification $M\mathcal{Q} \cong M \otimes_L L\mathcal{Q}$. Let τ be a generator for this action, where for example $\tau(a\varepsilon_{(i,\mu)}) = a^q\varepsilon_{(i,\mu)}$. Then τ acts on $M \otimes_L \Lambda$ as $a \otimes b \mapsto a^q \otimes b^q$. For example, suppose that $b \in \mathbb{F}_{q^{d_i}}$. Then

$$\tau(a \otimes b) = \tau\big(\sum_\mu ab^{q^\mu}\varepsilon_{(i,\mu)}\big) = \sum_\mu a^q b^{q^{\mu+1}}\varepsilon_{(i,\mu)} = a^q \otimes b^q. \tag{8.1.7}$$

In particular, the L-algebra automorphism $\mathbf{a}^{-1}\tau$ of $M \otimes_L \Lambda$ sends $a \otimes b$ to $a^q \otimes b$ and so we recover our original G action. Thus we deduce the following proposition from Corollary 4.1.4.

Proposition 8.1.3. *There is a bijection between the isomorphism classes of* Λ*-modules of dimension vector* α *and representations* Y *of* $M\mathcal{Q}$ *of dimension vector* $f^{-1}(\alpha)$ *such that* ${}^{\mathbf{a}}Y \cong {}^{\tau}Y$*, where* f *is again the canonical map from the* $\mathbf{a}$*-fixed points of the root lattice for* $\mathcal{Q}$ *to the root lattice for* Γ*.*

8.2 Kac's Theorem for Species

Let $L = \mathbb{F}_q$ be a finite field, Γ a valued quiver and denote by S the corresponding L-species of Γ. We write $I_\Gamma(\alpha, q)$ for the number of isomorphism classes of indecomposable representations of S of dimension vector α.

Theorem 8.2.1 (Hua). *The numbers $I_\Gamma(\alpha, q)$ are rational polynomials in q and are independent of the orientation of Γ.*

The proof can be found in [18].

We can now offer a more representation-theoretic proof of the following theorem, which generalises Kac's Theorem to representations of species over finite fields.

Theorem 8.2.2. *Let L and Γ be as above. Then the polynomials $I_\Gamma(\alpha, q)$ are non-zero if and only if α is a positive root of $\mathfrak{g}(\Gamma)$. Moreover, if α is a real root of $\mathfrak{g}(\Gamma)$ then $I_\Gamma(\alpha, q) = 1$.*

Proof.
Let Γ have symmetrisable GCM $C = D^{-1}B$, let S be the L-species of Γ and write Λ for the corresponding tensor algebra. Set t to be the lowest common multiple of the d_i and let M/L be a field extension of degree t. Then $M \otimes_L \Lambda$ is isomorphic to a path algebra $M\mathcal{Q}$ and as before, we can consider the two different actions of $G = \mathrm{Gal}(M/L)$ on $M\mathcal{Q}$ generated by $\mathbf{a}$ and τ.

We know that the isomorphism classes of indecomposable Λ-modules are in bijection with the isomorphism classes of representations Y for $M\mathcal{Q}$ such that ${}^{\mathbf{a}}Y \cong {}^{\tau}Y$ and Y is not the proper direct sum of two such representations. In particular, every ii-indecomposable Y for $(\mathcal{Q}, \mathbf{a})$ of dimension vector $f^{-1}(\alpha)$ defined over L gives rise to a Λ-indecomposable of dimension vector α.

Let X be a Λ-indecomposable of dimension vector α and let Y be an indecomposable summand of $M \otimes_L X$. Setting $\underline{\dim}\, Y = \beta$, we see that $\underline{\dim}\, M \otimes_L X = f^{-1}(\alpha) = r\sigma(\beta)$ for some r. Thus if β is an imaginary root of $\mathcal{Q}$, then $\alpha = rf(\sigma(\beta))$ is an imaginary root of Γ.

Conversely, if β is real, then Y is defined over L and so fixed by τ. Therefore $M \otimes_L X$ is an ii-indecomposable for $(\mathcal{Q}, \mathbf{a})$. Hence $r = 1$ and α is a root of Γ. This shows that all indecomposable Λ-modules have dimension vector a root of Γ and hence the polynomial $I_\Gamma(\alpha, q)$ is non-zero only if $\alpha \in \Delta(\mathfrak{g})_+$ (C.f. the argument in [18]).

Moreover, if α is real, then $\bar{\alpha} = 1$ and so $r = 1$. Therefore β is real and unique up to $\mathbf{a}$-orbit. Since there is a unique isomorphism class of ii-indecomposables for $(\mathcal{Q}, \mathbf{a})$ of dimension vector $f^{-1}(\alpha)$, we deduce that $I_\Gamma(\alpha, q) = 1$.

Finally, suppose that $\alpha \in \Delta(\Gamma)_+$. Then the polynomial $A\text{-}ii(\alpha, q)$ is non-zero and so for some finite field $L = \mathbb{F}_q$ with $q \equiv 1 \bmod t$ there exists an absolutely ii-indecomposable X for $(\mathcal{Q}, \mathbf{a})$ of dimension vector $f^{-1}(\alpha)$ defined over L. This representation corresponds to an indecomposable Λ-module of dimension vector α. Thus the polynomial $I_\Gamma(\alpha, q)$ cannot be zero.

■

Appendix

A Affine Schemes

In this appendix we review some of the theory of affine schemes and offer proofs of some of the theorems used in Chapter 5. These results are not given in their most general form, but will be sufficient to cover our needs.

Most of the theorems and proofs can be found in any good introduction to scheme theory, for example [6, 10, 13, 17, 33, 39].

A.1 Survey of Basic Results

Let A be a ring. The topological space $\operatorname{Spec} A$ is the set of prime ideals x of A endowed with the Zariski topology. A basis for this topology is given by the distinguished open sets $D(a) = \{x \not\ni a\}$ for $a \in A$. Moreover, this basis is closed under intersections. The closed sets are of the form $V(\mathfrak{a}) = \{x \supset \mathfrak{a}\}$ for some ideal $\mathfrak{a}$ of A, so that $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$. We note that $\operatorname{Spec} A$ is quasi-compact and is noetherian if and only if A is noetherian.

For two points $x, y \in \operatorname{Spec} A$, we say that x is a generalisation of y, and y a specialisation of x, if $y \in \overline{\{x\}}$. A point is called maximal if it has no proper generalisations.

We define the structure sheaf $\mathcal{O}$ on $\operatorname{Spec} A$ by setting $\mathcal{O}(D(a)) = A_a$ and more generally, for any open U we set

$$\mathcal{O}(U) := \varprojlim_{D(a)\subset U} A_a = \{(f_a) \in \prod A_a \mid f_a = f_b \in A_b \text{ whenever } D(b) \subset D(a)\}.$$

The stalk at a point x is given by $\mathcal{O}_x := \varinjlim_{a\notin x} A_a = A_x$, so we obtain a locally ringed space. The residue field $\kappa(x)$ at x is then the residue field of A_x.

An affine scheme is one of the form $X = (\operatorname{Spec} A, \mathcal{O}_A)$. Morphisms

$$f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$$

between affine schemes correspond to ring homomorphisms $\varphi : A \to B$. In this way, the category of affine schemes is isomorphic to the category of rings with arrows reversed.

A morphism $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ is of finite type (respectively finite) if the corresponding ring homomorphism $\phi : A \to B$ makes B into a finitely generated A-algebra (respectively a finitely generated A-module). The morphism f is dominant if its image is dense in X, which is if and only if the kernel of ϕ is contained in the nilradical of A.

We say that X is irreducible if it is irreducible as a topological space, which is equivalent to the nilradical of A being prime. In this case there is a unique maximal point, called the generic point. We also say that X is reduced if the nilradical is 0, and integral if it is both irreducible and reduced — that is, if A is an integral domain.

Any locally closed subset Y of X can be endowed with the structure of a reduced scheme as follows. Let V be the closure of Y in X, given by the radical ideal $\mathfrak{a}$. If we set $B = A/\mathfrak{a}$, then $(\operatorname{Spec} B, \mathcal{O}_B)$ is a closed reduced subscheme of X and we give Y the structure sheaf $\mathcal{O}_B|_Y$.

Now suppose that $f : X \to S$ and $g : Y \to S$ are morphisms of affine schemes, with X, Y and S given by rings A, B and R respectively. The fibred product of X and Y over S, denoted by $X \times_S Y$, is the affine scheme given by the ring $A \otimes_R B$. We then have the morphism $\pi_X : X \times_S Y \to X$ coming from the natural map $A \to A \otimes_R B$, and similarly for π_Y. The fibred product has the universal property that if $h : T \to S$ is another S-scheme and there are morphisms $\theta : T \to X$ and $\phi : T \to Y$ such that $f\theta = g\phi$, then there exists a unique morphism $\psi : T \to X \times_S Y$ such that $\pi_X \psi = \theta$ and $\pi_Y \psi = \phi$.

The fibred product is associative and $X \times_S S \cong X$. Also, finite type is preserved under base change. That is, if $f : X \to S$ is a morphism of finite type and $h : T \to S$ is any morphism, then the pull-back $g : X \times_S T \to T$ is of finite type. (For, if A is finitely generated over R, then $A \otimes_R B$ is finitely generated over B.) We sometimes write $X \otimes_R B$ for $X \times_{\operatorname{Spec} R} \operatorname{Spec} B$.

A special case of the fibred product is the fibre over a point.

Lemma A.1. *Let $f : X \to S$ be a morphism of affine schemes. Pick $s \in S$ and consider the natural morphism $\operatorname{Spec} \kappa(s) \to S$. Then the fibre of $f : X \to S$ over s is the scheme $X_s := X \times_S \operatorname{Spec} \kappa(s)$. This is a scheme over $\kappa(s)$ with underlying topological space homeomorphic to $f^{-1}(s)$*

Proof.
Let X and S be given by the rings A and R respectively and write $\varphi : R \to A$ for the ring homomorphism corresponding to f. Then the prime ideals of $A \otimes_R \kappa(s)$ correspond precisely with the prime ideals x of A such that $f(x) = s$.
■

Lemma A.2. *Let $f : X \to S$ be a morphism between affine schemes and suppose that S is irreducible with generic point η. Then the maximal points of X_η are all maximal in X.*

Proof.
Let x be maximal in X_η and suppose that ξ is a generalisation of x in X. Then $f(\xi)$ is a generalisation of $f(x) = \eta$, so $\xi \in f^{-1}(\eta)$ and hence $\xi = x$.
■

A.2 Chevalley's Theorem

We now prove Chevalley's Theorem for affine schemes.

Lemma A.3. *Let $R \to A$ be a ring homomorphism making A into a finitely generated R-algebra and assume that R is a domain. Let M be a finitely generated A-module. Then there exists $0 \neq r \in R$ such that M_r is a free R_r-module.*

Proof.
There exists a finite filtration $M = M_1 \supset M_2 \supset \cdots \supset M_m = 0$ such that $M_i/M_{i+1} \cong A/x_i$ for some prime ideals x_i. Since localisation preserves freeness and an extension of free modules is again free, it is enough to show the result for $M = A/x$ for some prime x. Replacing A by A/x we may assume that A is a domain and that $M = A$. Furthermore, for any non-zero $r \in R$ in the kernel of the map we have $A_r = 0$. Therefore we may also assume that $R \to A$ is injective.

Let K and L be the quotient fields of R and A respectively. Then the Noether Normalisation Lemma implies that there is a polynomial subalgebra $K[T_1, \ldots, T_d]$ of $A \otimes_R K$ over which $A \otimes_R K$ is integral. By clearing denominators, we may assume that each $T_i \in A$. We can now consider a finite set of generators for this extension, each of which satisfies an integral equation. If $0 \neq r \in R$ is a common denominator for the coefficients in these equations, then A_r is integral over $B = R_r[T]$.

We have a finite filtration $A_r = N_1 \supset \cdots \supset N_n = 0$ with $N_i/N_{i+1} \cong B/y_i$ for some primes y_i. If $y_i = 0$, then we know already that B is free over R_r. Otherwise $y_i \neq 0$ and so B/y_i has dimension strictly less than d. The result follows by induction on d.

■

Theorem A.4 (Chevalley). *Let $f : Y \to X$ be a morphism of finite type between affine noetherian schemes and let $Z \subset Y$ be a constructible subset. Then the set-theoretic image $f(Z)$ is constructible in X. In particular, $f(Y)$ is constructible.*

Proof.
Let $f : Y \to X$ correspond to the map $\phi : A \to B$ of noetherian rings, making

B into a finitely generated A-algebra. As in Lemma B.4 (with $G = 1$), we can express Z as a finite disjoint union of irreducible locally closed sets, and it is clearly enough to consider each of these in turn.

The closure of Z is again an affine scheme, so we may assume that Z is dense and open in Y. Also, since Y is noetherian, we can express Z as a finite union of distinguished open sets $D(b) = \operatorname{Spec} B_b$ and B_b is again a finitely generated A-algebra. Therefore we are reduced to the case when $Z = Y$ is irreducible.

We can replace X by the closure of $f(Y)$, which is again irreducible. We may also assume that X and Y are reduced, so $A \to B$ is now an inclusion of noetherian domains. The lemma above now shows that B_a is free over A_a for some $0 \neq a \in A$. This implies that $D(a)$ is contained in the image of f. For, let $x \not\ni a$ be a prime of A. Then B_x is again free over A_x and $xB_x \neq B_x$. Let y' be a maximal ideal of B_x containing xB_x. Then $y' \cap A_x = xA_x$ since this is the unique maximal ideal. Therefore the prime $y = y' \cap B$ satisfies $f(y) = x$.

Thus we have shown that $f(Y)$ contains a dense open subset of its closure. We are now done by induction on $\dim X$.

■

As a corollary of the proof we have

Corollary A.5. *Let $Z \subset X$ be a constructible subset of an affine noetherian scheme X. Then there exists an affine noetherian scheme Y and a morphism of finite type $f : Y \to X$ such that $f(Y) = Z$.*

Proof.

We can express Z as a finite disjoint union of irreducible and locally closed sets Z_i. Let $W_i = \overline{Z}_i$. These are again affine noetherian schemes and we have morphisms of finite type $W_i \to X$. Furthermore, we can express each Z_i as a finite union of distinguished open subsets $Z_{ij} \subset W_i$. Therefore each Z_{ij} is an affine noetherian scheme and we have morphisms of finite type $Z_{ij} \to W_i \to X$.

Write B_{ij} for the ring corresponding to Z_{ij} and set $B = \prod_{i,j} B_{ij}$. This is a noetherian ring which is finitely generated as an A-algebra via the obvious map

$A \to B$. Therefore we have a morphism of finite type $f : \operatorname{Spec} B \to X$ whose image is Z by construction.

■

A.3 Algebraic Schemes

Let K be a field and X an affine K-scheme — i.e. we have a structure morphism $f : X \to \operatorname{Spec} K$. We say that X is an affine algebraic K-scheme if f is of finite type.

Lemma A.6. *Let X be an affine algebraic K-scheme. If L/K is a field extension, then the morphism $f : X_L := X \otimes_K L \to X$ is dominant and open.*

Proof.
Let $X = \operatorname{Spec} A$, where A is a finitely generated K-algebra. We note first that f is open if and only if, for every point $y \in X_L$ and every neighbourhood U of y, the image $f(U)$ contains a neighbourhood of $x = f(y)$. This is equivalent to $f(U)$ containing all generalisations of x, or in other words $\operatorname{Spec} \mathcal{O}_{X,x} \subset f(U)$.

To see this, suppose that $f(U)$ contains all generalisations of x. By Corollary A.5 we have an affine algebraic scheme $Y = \operatorname{Spec} R$ and a morphism of finite type $g : Y \to X$ whose image is precisely $X - f(U)$. Then $R \otimes_A A_x = 0$, since primes of $R \otimes_A A_x$ correspond to primes y of R such that $g(y)$ is a generalisation of x. Since $A_x = \varinjlim_t A_t$ as t runs through $A - x$ we must have that $R \otimes_A A_t = 0$ for some such t. That is, $D(t)$ is a neighbourhood of x contained in $f(U)$. The converse is clear.

We can write L as the direct limit over all its finitely generated K-subalgebras, say $L = \varinjlim_\lambda L_\lambda$. Since $K \to L_\lambda$ is flat for all λ, so is $\phi_\lambda : A \to B_\lambda := A \otimes_K L_\lambda$. Therefore, for each prime y of B_λ, and setting $x = \phi_\lambda^{-1}(y)$, we have that $A_x \to (B_\lambda)_y$ is a flat morphism of local rings, and hence faithfully flat ([32], Corollary 4.A). Therefore the morphism $\operatorname{Spec}(B_\lambda)_y \to \operatorname{Spec} A_x$ is surjective for all y. We deduce that the morphism $f_\lambda : \operatorname{Spec} B_\lambda \to \operatorname{Spec} A$ is open.

Now consider $b \in B$. Then for some λ there exists $b_\lambda \in B_\lambda$ mapping to b. Thus the preimage of $D(b_\lambda)$ in $\operatorname{Spec} B$ is $D(b)$ and we show that $f(D(b)) = f_\lambda(D(b_\lambda))$, and hence is open.

Clearly $f(D(b)) \subset f_\lambda(D(b_\lambda))$ since there is a factorisation $A \to B_\lambda \to B$, so suppose that $x \in f_\lambda(D(b_\lambda))$. Then $B_\lambda \otimes_A \kappa(x) \cong L_\lambda \otimes_K \kappa(x)$ and similarly for $B \otimes_A \kappa(x)$. Since $K \to \kappa(x)$ is flat, the map $L_\lambda \otimes_K \kappa(x) \to L \otimes_K \kappa(x)$ is injective. Therefore $f^{-1}(x) \to f_\lambda^{-1}(x)$ is dominant and hence $f^{-1}(x)$ is non-empty.

Finally, since $A \to A \otimes_K L$ is injective, f is dominant.
■

Let X be an affine algebraic K-scheme. We say that X is geometrically irreducible if $X_L := X \otimes_K L$ is irreducible for all field extensions L/K. Similarly, X is geometrically reduced if X_L is reduced for all L/K.

Suppose that $X = \operatorname{Spec} A$, where A is a finitely generated K-algebra. Then X is geometrically irreducible if and only if $A \otimes_K L$ has prime nilradical for all L/K.

Lemma A.7. *Let $B = A/\operatorname{nil} A$. Then $\operatorname{Spec} A$ is geometrically irreducible if and only if B is a domain, say with quotient field M, and M/K is a primary field extension — that is, $\operatorname{nil}(M \otimes_K L)$ is prime for all L/K.*

Proof.
We have the short exact sequence

$$0 \to (\operatorname{nil} A) \otimes_K L \to A \otimes_K L \to B \otimes_K L \to 0$$

and $(\operatorname{nil} A) \otimes_K L$ is nilpotent. Therefore $\operatorname{nil}(A \otimes_K L)$ is prime if and only if $\operatorname{nil}(B \otimes_K L)$ is.

Similarly, $M \otimes_K L$ is a localisation of $B \otimes_K L$ and $\operatorname{nil}(B \otimes_K L) = (B \otimes_K L) \cap \operatorname{nil}(M \otimes_K L)$. Therefore $\operatorname{nil}(B \otimes_K L)$ is prime if and only if $\operatorname{nil}(M \otimes_K L)$ is.
■

We also have that X is geometrically reduced if and only if $\operatorname{nil}(A \otimes_K L) = 0$ for all L/K.

Lemma A.8. *The scheme* $\operatorname{Spec} A$ *is geometrically reduced if and only if it is reduced and the residue fields at each of the minimal primes of* A *belonging to 0 are separable over* K.

A field extension M/K is called separable if every finitely generated subextension M' of M is separably algebraic over a purely transcendental extension of K.

Proof.
Clearly we need A reduced. Also, since A is noetherian, its nilradical is the intersection of the minimal primes belonging to 0. Let x_i be these prime ideals and let $\kappa(x_i)$ be the residue field of A_{x_i}. Then the homomorphism $A \to \prod_i \kappa(x_i)$ is injective and so $A \otimes_K L$ embeds in $\prod_i \kappa(x_i) \otimes_K L$ for all L/K.

Now, by MacLane's Criterion ([4], Theorem 5.11), $\operatorname{Spec} M$ is geometrically reduced if and only if M/K is separable. Therefore, if each $\kappa(x_i)$ is separable, then $\prod_i \kappa(x_i) \otimes_K L$ is reduced for each L/K and so $\operatorname{Spec} A$ is geometrically reduced. Conversely, if $\operatorname{Spec} A$ is geometrically reduced, then any localisation of $A \otimes_K L$ is reduced, so in particular $A_{x_i} \otimes_K L$ is reduced. By the minimality condition, $x_i A_{x_i}$ is the unique prime ideal of A_{x_i} and hence is 0 since A_{x_i} is reduced. Therefore A_{x_i} is a domain and so the localisation $\kappa(x_i) \otimes_K L$ is reduced for all L/K. That is, $\kappa(x_i)/K$ is separable.
■

Lemma A.9. M/K *is primary if and only if* K *is separably algebraically closed in* M. *As a corollary, if* A *is a finitely generated* K*-algebra, then* $A \otimes_K L$ *is a domain for all* L/K *if and only if the quotient field* M *of* A *is separable over* K *and* K *is algebraically closed in* M.

Proof.
Suppose that M/K is primary and $x \in M - K$ is separable and algebraic over K with minimal polynomial f. Let $L = K[X]/(f)$. This is a separable extension of K and so $M \otimes_K L$ is reduced and irreducible, hence a domain. However, $X - x$ is a factor of f in $M[X]$, so $M[X]/(f)$ is not a domain. Contradiction.

Conversely, suppose that K is separably algebraically closed in M. We note that if $M \otimes_K L$ has prime nilradical for all algebraically closed fields L, then M/K will be primary. For, given any L, we have $\mathrm{nil}(M \otimes_K L) = (M \otimes_K L) \cap \mathrm{nil}(M \otimes_K \overline{L})$.

If L/K is separable and algebraic, then $L = \varinjlim_\lambda L_\lambda$ is a direct limit of its finite separable subextensions and $L_\lambda = K[X]/(f_\lambda)$ for some separable irreducible polynomial in $K[X]$. Each f_λ must remain irreducible in $M[X]$ and so $\mathrm{nil}(M \otimes_K L_\lambda)$ is prime. Hence $\mathrm{nil}(M \otimes_K L) = \varinjlim_\lambda (M \otimes_K L_\lambda)$ is prime.

In particular, let K_s be the separable closure of K in $\overline{K}$ and let N be the quotient field of $M \otimes_K K_s$. Then $M \otimes_K \overline{K}$ has prime nilradical if and only if $N \otimes_{K_s} \overline{K}$ does. Therefore we may assume that $K = K_s$ and $M = N$. In this case, $\overline{K}/K$ will be p-radical, where $p > 0$ is the characteristic of K.

Let $x = \sum_i x_i \otimes \lambda_i$ and $y = \sum_j y_j \otimes \lambda_j$ be elements of $M \otimes_K \overline{K}$ such that $(xy)^n = 0$. Let q be a large enough power of p such that λ_i^q and λ_j^q are in K and $(xy)^q = 0$. Then $(\sum_i x_i^q \lambda_i^q)(\sum_j y_j^q \lambda_j^q) = 0$ in M and so without loss of generality $\sum_i x_i^q \lambda_i^q = 0$. That is, $x^q = 0$ in $M \otimes_K \overline{K}$ and $\mathrm{nil}(M \otimes_K \overline{K})$ is prime.

Finally, by considering the quotient field of $M \otimes_K \overline{K}$ modulo its nilradical, we may assume that K is algebraically closed. Then, by MacLane's Criterion, every extension of K is separable. As before, it is enough to consider finitely generated extensions L/K, which must be separably algebraic over some purely transcendental extension $K(T_1 \dots, T_d)$ of K. Hence there exists an irreducible polynomial $f \in K[T, X]$ such that $L = K(T)[X]/(f)$. Now $M \otimes_K K[T, X]/(f) \cong M[T, X]/(f)$ so by Gauss's Lemma ([28], Chapter IV, Theorem 2.1), if we can show f remains irreducible in $M[T, X]$, then $M(T)[X]/(f)$ will be a field. Since $M \otimes_K L \subset M(T)[X]/(f)$, we deduce that M/K is primary.

Let $f \in K[T_1, \dots, T_d, X]$. Write $T_0 = X$ and consider the K-algebra homomorphism $\phi : M[T] \to M[Y]$ given by $T_i \mapsto Y^{N^i}$ for some $N \gg 0$. Suppose that $f = gh \in M[T]$. Since $\phi(f) = \phi(g)\phi(h)$, we may assume that $\phi(g)$ and $\phi(h)$ have leading coefficients in K. Therefore all the coefficients of $\phi(g)$ and $\phi(h)$ are algebraic over K, hence lie in K, and so $\phi(g), \phi(h) \in K[Y]$. However, ϕ is injective on polynomials of degree less than N, so $g, h \in K[T]$. Therefore, if f is irreducible in $K[T, X]$, then it remains irreducible over $M[T, X]$.

■

We can now prove

Theorem A.10. *Let X be an affine algebraic K-scheme. Then there exists a finite field extension L/K such that every irreducible component of $(X_L)_{\mathrm{red}}$ is geometrically integral.*

We prove this in two parts.

Proposition A.11. *Let X be an irreducible affine algebraic K-scheme with generic point ξ. Let E be the separable algebraic closure of K in $\kappa(\xi)$ and L/K a Galois extension containing E. Then X_L has $[E : K]$ irreducible components, each of which is geometrically irreducible.*

Proof.

Since X is of finite type over K, $\kappa(\xi)/K$ is a finitely generated field extension and hence E/K is finite. Therefore we may take L/K to be finite. Since $X_L \to X$ is dominant and open, the maximal points of X_L correspond to the minimal primes of $\kappa(\xi) \otimes_K L$.

Let $E = K[T]/(f)$ for some separable irreducible polynomial f of degree $m = [E : K]$ and consider $\kappa(\xi) \otimes_K L = \kappa(\xi) \otimes_E (E \otimes_K L)$. We know that $E \otimes_K L$ is isomorphic to the product of m copies of L and that $\kappa(x)/E$ is primary. Therefore $\operatorname{Spec} \kappa(\xi)$ is geometrically irreducible over E, so in particular each $\operatorname{Spec} \kappa(\xi) \otimes_E L$ is geometrically irreducible.

■

Proposition A.12. *Let X be an affine algebraic K-scheme. Then there exists a finite extension L/K such that $(X_L)_{\mathrm{red}}$ is geometrically reduced.*

Proof.

Let $X = \operatorname{Spec} A$ and let $\mathfrak{n} = \mathrm{nil}(A \otimes_K \overline{K})$. This is finitely generated over $\overline{K}$ and so all these generators lie in some finite extension L of K. Hence there exists a finite subextension L/K such that if $\mathfrak{n}_L = \mathrm{nil}(A \otimes_K L)$, then $\mathfrak{n} = \mathfrak{n}_L \otimes_L \overline{K}$. Since every extension of $\overline{K}$ is separable, $(X_L)_{\mathrm{red}} = \mathrm{Spec}(A \otimes_K L/\mathrm{n}_L)$ is geometrically reduced.

■

Theorem A.10 follows immediately from these two propositions.

A.4 Arithmetic Schemes

According to [10], an arithmetic scheme is a scheme of finite type over $\mathbb{Z}$. Here we shall consider a slightly more general notion. We fix an affine noetherian integral scheme S and let X be an affine S-scheme of finite type such that the structure morphism $f : X \to S$ is dominant. We set $S = \mathrm{Spec}\, R$ and $X = \mathrm{Spec}\, A$, so that R is a noetherian domain, $R \to A$ is injective and A is a finitely generated R-algebra. We denote the generic point of S by η.

We show that certain properties of X_η hold for all X_s with s in an open neighbourhood of η.

Proposition A.13. *Let Z be a constructible subset of X and suppose that Z_η is dense in X_η. Then Z_s is dense in X_s for all s in an open neighbourhood of η.*

Proof.
Let $X_1 \dots, X_m$ be the closures in X of the irreducible components of X_η. These are irreducible components of X by Lemma A.2. Also, if Y is any other irreducible component of X then the morphism $Y \to S$ does not contain η. Since its image is constructible by Chevalley's Theorem, it must be contained in a proper closed subset and hence $Y_s = \emptyset$ for all s in an open neighbourhood of η. In particular, we can take an open neighbourhood of η over which X_s is the union of the $(X_i)_s$.

Since $Z \cap X_i$ is dense in X_i, it is enough to show that this holds over an open neighbourhood of η. That is, we may assume that X is irreducible. Clearly we may also take X to be reduced, so integral. Now Z is constructible and dense in X, so it contains a distinguished open set $D(a)$ for some $0 \neq a \in A$. Therefore it is enough to show that $D(a)_s$ is dense in X_s for all s in an open neighbourhood of η — i.e. the map $A \otimes_R \kappa(s) \to A_a \otimes_R \kappa(s)$ is injective.

Consider the short exact sequence $0 \to A \to A_a \to A_a/A \to 0$. If we can show that $(A_a/A)_r$ is a flat R_r-module for some $0 \neq r \in R$, then $\mathrm{Tor}_1^{R_r}((A_a/A)_r, -)$ vanishes and the result follows.

Let $A\frac{1}{a^m}/A$ be the submodule of A_a/A generated by $\frac{1}{a^m}$. We have the natural inclusions $A\frac{1}{a^m}/A \hookrightarrow A\frac{1}{a^{m+1}}/A$ and $A_a/A = \varinjlim_m A\frac{1}{a^m}/A$. Also, each quotient is isomorphic to $A\frac{1}{a}/A$, which is a finitely generated A-module. Therefore, by Lemma A.3, there exists some $0 \neq r \in R$ such that $(A\frac{1}{a}/A)_r$, and so each $(A\frac{1}{a^m}/A)_r$, is free over R_r. Hence $(A_a/A)_r$ is flat over R_r.
■

Theorem A.14. *There exists an open neighbourhood of η over which $s \mapsto \dim X_s$ is constant.*

Proof.
As in the proof of Proposition A.13, we may assume that each irreducible component of X meets $f^{-1}(\eta)$. Since $\dim X_s = \max_i \dim(X_i)_s$, it is enough to prove the result when X is irreducible. We may also assume that X is reduced, so integral.

The construction in the proof of Lemma A.3 shows that for some $0 \neq r \in R$, A_r is finite over a polynomial subalgebra $R_r[T_1, \ldots, T_d]$. Therefore, for all primes s of R not containing r, A_s is integral over $R_s[T]$. Finally $\kappa(s) \otimes_R A$ is integral over the polynomial subalgebra $\kappa(s)[T]$ and so $\dim X_s = d$.
■

Lemma A.15. *Let A be a noetherian domain with quotient field K and let $F \in A[T_1, \ldots, T_d]$ be a non-constant polynomial. If F_K is geometrically irreducible, then there exists $0 \neq a \in A$ such that $F_{\kappa(x)}$ is geometrically irreducible for all primes $x \not\ni a$.*

N.B. $F_{\kappa(x)}$ is the image of F in $\kappa(x)[T]$ and this is called geometrically irreducible if it remains irreducible over all $L/\kappa(x)$.

Proof.
Let $F = \sum_\alpha c_\alpha T^\alpha$, using multi-indices $\alpha \in \mathbb{N}^d$. Let F have degree r, so that $\operatorname{ht}\alpha = \sum_i \alpha_i \leq r$ for all $c_\alpha \neq 0$. We can take $0 \neq a \in A$ such that for all primes $x \not\ni a$, each non-zero c_α remains non-zero in $\kappa(x)$ and hence $F_{\kappa(x)}$ has degree r.

Let (p, q) be a pair of positive integers summing to r. For all multi-indices β and γ with $\operatorname{ht}\beta \leq p$ and $\operatorname{ht}\gamma \leq q$ we introduce the indeterminates P_β and Q_γ. Let B be the polynomial A-algebra generated by these indeterminates. For each α consider the polynomial $G_\alpha = c_\alpha - \sum_{\beta+\gamma=\alpha} P_\beta Q_\gamma$ and let $\mathfrak{b}$ be the ideal of B generated by the G_α.

Let $L = \kappa(x)$ for some prime $x \not\ni a$. Then to say that there is a factorisation $F_L = F_1 F_2$ over $\overline{L}[T]$ with F_1 and F_2 of degrees p and q respectively is to say that there exists a solution in $\overline{L}$ to the system of equations $(G_\alpha)_L = 0$ for all α. Equivalently, the affine scheme $\operatorname{Spec} B/\mathfrak{b}$ has an $\overline{L}$-valued point.

Thus if F_K is geometrically irreducible, then no such point over $\overline{K}$ exists. That is, the fibre $V(\mathfrak{b})_\xi$ is empty, where ξ is the generic point of $\operatorname{Spec} A$. Therefore, by Chevalley's Theorem, the same is true over an open neighbourhood of η and $F_{\kappa(x)}$ has no factoristion into two polynomials of degrees p and q. Since there are only finitely many such pairs (p, q), the result follows.
■

Theorem A.16. *If X_η is geometrically integral, then so is X_s for s in an open neighbourhood of η.*

Proof.
We first note that the fibre X_s is geometrically integral if and only if it is integral and the residue field at the maximal point is primary and separable (i.e. regular). Therefore, by Proposition A.13, it is enough to prove the result for a distinguished open subset of X.

Let $R \hookrightarrow A$ be the inculsion of noetherian domains and let K and L be their respective quotient fields. Then L is finite and separable over $K(T_1, \ldots, T_d)$, so there exists a separable polynomial $F \in K(T)[t]$ such that $L = K(T)[t]/(F)$. By clearing denominators, we may assume that $F \in K[T,t]$ and so $F \in R_r[T,t]$ for some $0 \neq r \in R$. Set $B = R_r[T,t]/(F)$ and note that this is a finitely generated domain over R with quotient field L.

Taking a finite set of generators for B over A, we see that $B \subset A_a$ for some $0 \neq a \in A$. Similarly, since A_a is finitely generated over B, we see that for some $0 \neq b \in B$, $B_b = A_{ab}$. Therefore it is enough to prove the result for B_b.

We know that F_K is geometrically irreducible, so $F_{\kappa(s)}$ is geometrically irreducible for all s in an open neighbourhood of η. Hence $\kappa(s) \otimes_R B = \kappa(s)[T,t]/(F_{\kappa(s)})$ is geometrically integral. Also, by Proposition A.13, $D(b)_s$ is dense in B_s over a non-empty subset of $\operatorname{Spec} R$. Combining these, we see that $D(b)_s$ is geometrically integral for all s in an open neighbourhood of η.

■

B Affine Algebraic Group Actions

Let K be a fixed algebraically closed field.

An affine algebraic group G is an affine variety and a group such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are both morphisms of varieties. This is equivalent to saying that the K-algebra representing G has the structure of a Hopf algebra ([21], Section 7.6 or [41], Section 1.4).

Let X be an affine variety on which G acts as a group. We say that G acts morphically on X if the map $G \times X \to X$ is a morphism of varieties. We shall only consider such group actions. In terms of algebras, if $G = \operatorname{Spec} H$ and $X = \operatorname{Spec} A$, then A has the structure of a comodule for the Hopf algebra H (see [41], Section 3.2).

We record the following two results on morphisms of varieties. Their proofs can be found in [33], Section I.8. The local dimension $\dim_x X$ at a point $x \in X$ is the maximum dimension of any irreducible component of X containing x.

Lemma B.1. *Let $f : X \to Y$ be a dominant morphism of irreducible varieties. Then any irreducible component of a fibre has dimension at least* $\dim X - \dim Y$*. Moreover, there exists a non-empty open U in Y over which every fibre has pure dimension* $\dim X - \dim Y$*.*

Lemma B.2 (Upper semicontinuity of fibre dimension). *Let $f : X \to Y$ be a morphism of varieties. Then the function $x \mapsto \dim_x f^{-1}(f(x))$ is upper semicontinuous.*

Lemma B.3. *Let G act on X. Then the orbits are locally closed and*

$$\dim G = \dim G \cdot x + \dim \mathrm{Stab}_G(x)$$

for any $x \in X$.

Proof.
Let $x \in X$ and consider the morphism $G \to X$ sending g to $g \cdot x$. This has as image the orbit $G \cdot x$ of x, which is constructible by Chevalley's Theorem. Therefore $G \cdot x$ contains a dense open subset U of its closure. Now $G \cdot x$ must equal $G \cdot U$ and this latter set is again open in its closure, since it is the union over all $g \in G$ of the sets $g \cdot U$. That is, $G \cdot x$ is locally closed.

The fibre of $G \to X$ over $g \cdot x$ is $g\mathrm{Stab}_G(x)g^{-1}$, which is isomorphic to $\mathrm{Stab}_G(x)$ and so of the same dimension. Therefore $\dim G = \dim G \cdot x + \dim \mathrm{Stab}_G(x)$ by Lemma B.1.
■

Lemma B.4. *Let G act on X and let Z be a constructible subset of X, stable under G. Then we can express Z as a finite disjoint union of G-stable, irreducible and locally closed subsets of X.*

Proof.
Let $Z = W_1 \cup \cdots \cup W_m$ be the decomposition of Z into irreducible components. We note that each W_i is G-stable, since $G \cdot W_i$ is again irreducible, contained in Z and contains W_i.

Now consider $W_i - \bigcup_{j \neq i} W_j$. This is again G-stable, irreducible and constructible, so contains a G-stable dense open subset U_i of its closure. For, it must contain a dense open subset U_i of its closure and then we can replace U_i by $G \cdot U_i$. Therefore, if we set $V = Z - \bigcup_i U_i$, then we can write Z as a disjoint union of the U_i and V. Each of these is G-stable, the U_i are irreducible and locally closed and V has strictly smaller dimension than Z.

The proof follows by induction on $\dim Z$.

■

Again, consider the action of G on X and let Z be a G-stable constructible subset of X. For $d \geq 0$ we define the sets

$$Z_{(d)} := \{z \in Z \mid \dim G \cdot z = d\}.$$

These are also G-stable and constructible.

To see this, consider the set

$$V' := \{(g, x) \in G \times X \mid g \cdot x = x\}$$

and let $\pi : V' \twoheadrightarrow X$ be the projection onto the second co-ordinate. By Lemma B.2 we know that the set

$$\{(g, x) \in V' \mid \dim \pi^{-1}(x) \geq d\}$$

is closed and so using the morphism $X \to V'$, $x \mapsto (1, x)$, we deduce that

$$\{x \in X \mid \dim \pi^{-1}(x) \geq d\}$$

is also closed. It follows that

$$\{x \in X \mid \dim \pi^{-1}(x) = d\}$$

is locally closed. Finally (using Lemma B.3) $Z_{(d)}$ is the intersection

$$Z_{(d)} = Z \cap \{x \in X \mid \dim \pi^{-1}(x) = \dim G - d\}.$$

The number of parameters for G acting on Z is defined to be

$$\dim_G Z := \max_d \{\dim Z_{(d)} - d\}$$

and the number of top-dimensional families of orbits is

$$\operatorname{top}_G Z := \sum_{\dim Z_{(d)} = d + \dim_G Z} \operatorname{top} Z_{(d)}.$$

Proposition B.5. *Keeping the same notation, let*

$$V := \{(g, z) \in G \times Z \mid g \cdot z = z\} \subset V'.$$

This is a constructible set and $\dim V = \dim G + \dim_G Z$*. Moreover, if each stabiliser* $\mathrm{Stab}_G(z)$ *for* $z \in Z$ *is irreducible, then* $\operatorname{top} V = \operatorname{top}_G Z$*.*

Proof.

Again, let π be the projection $V' \to X$ and set $V_{(d)} := \pi^{-1}(Z_{(d)})$, so that each fibre of $\pi : V_{(d)} \to Z_{(d)}$ has dimension $\dim G - d$. Since V and Z are the disjoint unions of the $V_{(d)}$ and $Z_{(d)}$ respectively, it is enough to show that $\dim V_{(d)} = \dim G + \dim Z_{(d)}$, and that $\operatorname{top} V_{(d)} = \operatorname{top} Z_{(d)}$ if each fibre is irreducible.

Therefore we may assume that each fibre over Z has constant dimension $\dim G - d$. Using Lemma B.3 we can decompose Z into a finite disjoint union of G-stable irreducible locally closed subsets of X, say $Z = Z_1 \cup \cdots \cup Z_m$. Set $V_i := \pi^{-1}(Z_i)$. Then V is the disjoint union of the V_i, each of which is locally closed in V, and the fibres of the morphism $\pi : V_i \to Z_i$ all have dimension $\dim G - d$. Moreover,

$$\dim_G Z = \dim Z - d = \max_i \{\dim Z_i\} - d, \qquad \dim V = \max_i \{\dim V_i\}$$

and

$$\operatorname{top}_G Z = \operatorname{top} Z = \sum_{\dim Z_i = \dim Z} \operatorname{top} Z_i, \qquad \operatorname{top} V = \sum_{\dim V_i = \dim V} \operatorname{top} V_i.$$

Thus we may further assume that Z is irreducible and locally closed.

That is, we have a G-stable, irreducible and locally closed set Z and a surjective morphism $\pi : V \to Z$ whose fibres each have dimension $\dim G - d$. Therefore, by Lemma B.1, $\dim V = \dim Z + \dim G - d$.

Now suppose that each fibre is irreducible. Then V is also irreducible. For, suppose that $V = V_1 \cup \cdots V_m$ is the decomposition of V into its irreducible components and let $\pi_i : V_i \to Z$ be the restriction of π. Let $d_i(z) = \dim \pi_i^{-1}(z)$ so that $\max_i\{d_i(z)\} = \dim G - d$ for each z. By Lemmas B.1 and B.2, each d_i is upper semicontinuous and attains its minimal value on an open subset of Z. Therefore, since Z is irreducible, $d_i \equiv \dim G - d$ for some i.

For this i we have that $\pi_i^{-1}(z)$ is a closed subvariety of $\pi^{-1}(z)$ of the same dimension, and since the fibre is irreducible they must coincide. We conclude that $V = V_i$ is irreducible. (This last argument is taken from [16], Theorem 11.14.)
■

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